

# Existence of solutions to a general geometric elliptic variational problem

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## Abstract

We consider the problem of minimising an inhomogeneous anisotropic elliptic functional in a class of closed  $m$  dimensional subsets of  $\mathbb{R}^n$  which is stable under taking smooth deformations homotopic to identity and under local Hausdorff limits. We prove that the minimiser exists inside the class and is an  $(\mathcal{H}^m, m)$  rectifiable set in the sense of Federer. The class of competitors encodes a notion of spanning a boundary. We admit unrectifiable and non-compact competitors and boundaries, and we make no restrictions on dimension  $m$  and co-dimension  $n - m$  other than  $1 \leq m < n$ . An important tool for the proof is a novel smooth deformation theorem. The skeleton of the proof and the main ideas originate from Almgren's 1968 paper. In the end we show that classes of sets spanning some closed set  $B$  in homological and cohomological sense satisfy our axioms.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Notation</b>	<b>4</b>
<b>3</b>	<b>Statement of the main result</b>	<b>5</b>
<b>4</b>	<b>Unrectifiable sets under submersions</b>	<b>7</b>
<b>5</b>	<b>Smooth almost retraction of <math>\mathbb{R}^n</math> onto a cube</b>	<b>13</b>
<b>6</b>	<b>Central projections</b>	<b>17</b>
<b>7</b>	<b>Smooth deformation theorem</b>	<b>21</b>
<b>8</b>	<b>Slicing varifolds by continuously differentiable functions</b>	<b>36</b>
<b>9</b>	<b>Density ratio bounds</b>	<b>40</b>
<b>10</b>	<b>Rectifiability of the support of the limit varifold</b>	<b>47</b>
<b>11</b>	<b>Unit density of the limit varifold</b>	<b>52</b>
<b>12</b>	<b>An example of a good class: homological spanning</b>	<b>65</b>

## 1 Introduction

The Plateau problem is about finding a minimiser of the area amongst the surfaces which span a given boundary. The notions of “area”, “surface”, and “spanning a boundary” of course need to be precisely defined so this problem actually has many different incarnations. Its history spans over two centuries and we have no intention of enumerating numerous prominent developments in this field. For the presentation of the classical formulations and solutions, their drawbacks, and the modern reformulations of the problem we refer the reader to the excellent expository articles by David [Dav14] as well as by Harrison and Pugh [HP15, HP16d]. These sources contain also an extensive list of references. We shall focus mainly on comparison of our methods and results with the ones used in the papers published in the last years.

In this article we deal with an abstract Plateau’s problem which encompasses, e.g., the formulation of Adams and Reifenberg; cf. [Rei60]. The notion of “area” of some competitor  $S$  is replaced by the value of a functional  $\Phi_F$  on  $S$  which is defined by integrating an *elliptic* integrand  $F : \mathbf{R}^n \times \mathbf{G}(n, m) \rightarrow [0, \infty]$  with respect to the Hausdorff measure  $\mathcal{H}^m$  over  $S$  – if  $S$  is  $(\mathcal{H}^m, m)$  rectifiable, then we feed  $F$  with pairs  $(x, T)$ , where  $x \in S$  and  $T$  is the approximate tangent plane to  $S$  at  $x$ . The integrand  $F$  provides an inhomogeneous (depending on the location) and anisotropic (depending on the tangent direction) weight for the Hausdorff measure. If  $F(x, T) = 1$  for all  $(x, T) \in \mathbf{R}^n \times \mathbf{G}(n, m)$ , then we call  $F$  the *area integrand*. Ellipticity means roughly that a flat  $m$ -dimensional disc  $D$  minimises  $\Phi_F$  amongst surfaces that cannot be retracted onto the boundary  $(m - 1)$ -dimensional sphere  $\partial D$ ; see 3.13. It can be seen as a geometric counterpart of quasi-convexity; see [Mor66]. The “surfaces” and “boundaries” are, in our case, quite arbitrary closed subsets of  $\mathbf{R}^n$  – not necessarily rectifiable nor compact. The notion of “spanning a boundary” does not appear at all. Our main theorem (see 3.16) provides existence of an  $(\mathcal{H}^m, m)$  rectifiable set which minimises  $\Phi_F$  (with  $F$  elliptic) inside an axiomatically defined class of competitors (see 3.4).

Section 3 contains all the definitions and the precise statement of the main theorem. In section 12 we show that naturally defined (using Čech homology and cohomology) classes of sets spanning a given boundary (which may be an arbitrary closed set in  $\mathbf{R}^n$ ) satisfy our axioms.

Similar results were obtained recently by other authors. Harrison [Har14] suggested a new formulation of the problem, defined spanning employing the linking number, and used differential chains, developed earlier in [Har15], to find minimisers of the Hausdorff measure in co-dimension one. Harrison and Pugh [HP16c, HP16b] prove existence of minimisers for the area integrand in arbitrary dimension and co-dimension using Čech cohomology to define spanning. The same authors prove existence of solutions to an inhomogeneous and anisotropic Plateau’s problem in [HP16a].

De Lellis, Ghiraldin, and Maggi [DLGM17] formulated the problem abstractly and showed existence of minimisers of the  $m$ -dimensional Hausdorff measure in any family of subsets of  $\mathbf{R}^{m+1}$  containing enough competitors; see [DLGM17, Definition 1]. Later De Philippis, De Rosa, and Ghiraldin [DPDRG16] generalised this result to any co-dimension assuming, roughly, that the set of competitors is closed under taking defor-

mations which are uniform limits of maps  $\mathcal{C}^1$  isotopic to identity; see [DPDRG16, Definition 1.2]. After that De Lellis, De Rosa, and Ghiraldin [DDG16a] obtained also minimisers for an inhomogeneous and anisotropic problem in co-dimension one. These works all consider axiomatically defined families of competitors, which include, e.g., sets that span a boundary in the sense of Harrison and *sliding* competitors of David [Dav14]. However, in the latter case the results of [DLGM17, DPDRG16, DDG16a] do not ensure that the minimiser is a sliding deformation of the initial competitor.

The origin of our project was a mini-course that we conducted a few years ago. We undertook the effort to understand Almgren’s existence result presented on the first 30 pages of [Alm68]. Enlightened by his brilliant ideas we decided to present his approach to the Plateau problem in a sequence of lectures. The present manuscript was, at first, meant to serve as lecture notes for the mini-course but, with time, it grew into a full-fledged research paper containing new results.

The skeleton of the proof is the same as in [Alm68] and our proofs of the intermediate steps are quite similar to Almgren’s but we also divert from [Alm68] in many places. First of all we separated the abstract existence result from the application to a specific class of sets which homologically span a given boundary. Actually, in the definition of the *good class* of competitors, we do not use any notion of “spanning a boundary” – we only assume the class is closed under local Hausdorff convergence, and under taking images of sets with respect to certain *admissible* deformations; see 3.4. Moreover, we make no use of currents, flat chains, or  $G$ -varifolds in this paper. Second, we filled up most of the details that Almgren left to the reader. In particular, we had to develop a new *smooth* deformation theorem, which might be of separate interest (see 7.13) and we proved a perturbation lemma (see 4.3) which allows to show rectifiability of minimisers. Third, we improved the main theorem by allowing for non-compact and unrectifiable competitors and boundaries.

Our deformation theorem 7.13 takes some  $m$  dimensional sets  $\Sigma_1, \dots, \Sigma_l$  and a finite subfamily  $\mathcal{A}$  of dyadic Whitney cubes and provides a  $\mathcal{C}^\infty$  smooth homotopy  $f : [0, 1] \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  between the identity and a map which deforms some neighbourhood  $G$  of  $\bigcup_{i=1}^l \Sigma_i \cap \bigcup \mathcal{A}$  onto an  $m$  dimensional skeleton of  $\mathcal{A}$ . Furthermore, for each  $t \in [0, 1]$  the map  $f(t, \cdot)$  equals the identity outside some neighbourhood of  $\bigcup \mathcal{A}$ . The main novelty with respect to well known constructions of this sort is that  $f$  is  $\mathcal{C}^\infty$  smooth. This is important for two reasons. First, the push-forward by  $f$  defines a *continuous* map  $f_\# : \mathbf{V}_m(\mathbf{R}^n) \rightarrow \mathbf{V}_m(\mathbf{R}^n)$  on the space of  $m$  dimensional varifolds in  $\mathbf{R}^n$ . This allows to transfer certain estimates valid for the limit varifold (which, a priori, is not a competitor) onto elements of the minimising sequence; see, e.g., 9.1. Second, since the image of  $G$  under  $f(1, \cdot)$  is  $m$  dimensional, we may use a perturbation argument (see 4.3) to find another smooth map which is arbitrarily close to  $f(1, \cdot)$  in  $\mathcal{C}^1$  topology and almost kills the measure of the unrectifiable part of  $\Sigma_i \cap \bigcup \mathcal{A}$ . This allows to show that the minimiser coming from the main theorem 3.16 is  $(\mathcal{H}^m, m)$  rectifiable.

In contrast to the classical Federer–Fleming deformation theorem [Fed69, 4.2.9] and Almgren’s deformation theorem [Alm86, Chapter 1] ours works for quite arbitrary sets  $\Sigma$  in  $\mathbf{R}^n$  which may not carry the structure of a rectifiable current. It differs also from a similar result of David and Semmes [DS00, Theorem 3.1] because we perform the deformation inside Whitney cubes of varying sizes and, in case  $\Sigma$  are  $(\mathcal{H}^m, m)$  rectifiable, we provide estimates on the  $\mathcal{H}^{m+1}$  measure of the whole deformation, i.e., on  $\mathcal{H}^{m+1}(f[(0, 1) \times \Sigma])$ . Moreover, our theorem is tailored especially for the use with varifolds which might not be

rectifiable, so we actually prove estimates not on the  $\mathcal{H}^m$  measure of  $f(1, \cdot)[\Sigma]$  but rather on the integral  $\int_{\Sigma} \|Df(1, \cdot)\|^m d\mathcal{H}^m$ .

In the course of the proof of the main theorem we try to mimic, as much as possible, Almgren's original ideas. In particular, rectifiability of the minimiser is proven employing the deformation theorem together with a perturbation argument based on the Besicovitch–Federer projection theorem; see section 10. This point of the proof seems to make a lot of trouble in other works. De Lellis, Ghiraldin, and Maggi in [DLGM17] and by De Philippis, De Rosa, and Ghiraldin in [DPDRG16] used the famous Preiss' rectifiability theorem [Pre87]. De Lellis, De Rosa, and Ghiraldin in [DDG16a] employed the theory of Caccioppoli sets, which is possible in co-dimension one. The first author in [Fan13] used a very complex construction of Feuvrier [Feu12] to modify a minimising sequence into a sequence consisting of quasi-minimal sets; cf. [Alm76]. Harrison and Pugh [HP16b, HP16a] use a different method to modify the sequence and obtain, so called, *Reifenberg regular sequence*. Following Almgren, we were able to avoid all of this.

In [DDG16a] the authors announced another result concerning existence of solutions to an anisotropic Plateau problem in arbitrary co-dimension. As a first step towards this result De Philippis, De Rosa, and Ghiraldin [DDG16b] acquired a sufficient and necessary condition (the *atomic condition*) on the integrand so that varifolds whose first variation with respect to such  $F$  induces a Radon measure are rectifiable. Later, De Rosa [De 16] showed that if  $F$  satisfies the atomic condition, then an  $F$ -minimising sequence of varifolds with density uniformly bounded from below contains a sub-sequence converging to a rectifiable varifold.

## 2 Notation

In principle we shall follow the notation of Federer; see [Fed69, pp. 669–671]. However, we will use the standard abbreviations for intervals in  $\mathbf{R}$ , i.e.,  $(a, b) = \{t \in \mathbf{R} : a < t < b\}$ ,  $[a, b) = \{t \in \mathbf{R} : a \leq t < b\}$  etc. We reserve the symbol  $I = [0, 1]$  for the closed unit interval. We will also write  $\{x \in X : P(x)\}$  rather than  $X \cap \{x : P(x)\}$  to denote the set of those  $x \in X$  which satisfy predicate  $P$ . For the identity map on some set  $X$  we use the symbol  $\text{id}_X : X \rightarrow X$  and the characteristic function of  $X$  is denoted  $\mathbb{1}_X$  and is defined by  $\mathbb{1}_X(x) = 1$  if  $x \in X$  and  $\mathbb{1}_X(x) = 0$  if  $x \notin X$ . If  $U \subseteq \mathbf{R}^m$  and  $V \subseteq \mathbf{R}^n$ , the set of maps  $f : U \rightarrow V$  with continuous  $k^{\text{th}}$  order derivatives is denoted by  $\mathcal{C}^k(U, V)$ . If  $f \in \mathcal{C}^k(U, V)$ , we say that  $f$  is of *class*  $\mathcal{C}^k$ .

Concerning varifolds we shall follow Allard's notation; see [All72]. In particular, if  $U \subset \mathbf{R}^n$  is open, we write  $\mathbf{V}_k(U)$ ,  $\mathbf{IV}_k(U)$ , and  $\mathbf{RV}_k(U)$  for the space of  $k$  dimensional varifolds, integral varifolds, and rectifiable varifolds in  $U$  following the definitions [All72, 3.1, 3.5]. Also  $\text{VarTan}(V, x)$  shall denote the set of varifold tangents as defined in [All72, 3.4].

We recall some notation of Federer. As in [Fed69, 2.2.6] we use the symbol  $\mathcal{P}$  to denote the set of positive integers. The symbols  $\mathbf{U}(a, r)$  and  $\mathbf{B}(a, r)$  denote respectively the open and closed ball with centre  $a$  and radius  $r$ ; see [Fed69, 2.8.1]. We use the notation  $\tau_a$  and  $\mu_s$  for the translation by  $a \in \mathbf{R}^n$  and the homothety with ratio  $s \in \mathbf{R}$  respectively; see [Fed69, 2.7.16, 4.2.8]. For the Hausdorff metric on compact subsets of  $\mathbf{R}^n$  we write  $d_{\mathcal{H}}$  and for the  $k$  dimensional Hausdorff measure  $\mathcal{H}^k$ . The scalar product of  $u, v \in \mathbf{R}^n$  is denoted  $u \bullet v$ . The space of maps  $p \in \text{Hom}(\mathbf{R}^n, \mathbf{R}^m)$  such that  $p^*u \bullet p^*v = u \bullet v$  for all

$u, v \in \mathbf{R}^m$  (i.e.  $p$  is an orthogonal projection) is denoted  $\mathbf{O}^*(n, m)$ ; see [Fed69, 1.7.4].

Following [Alm68] and [Alm00] if  $S \in \mathbf{G}(n, m)$  is an  $m$  dimensional linear subspace of  $\mathbf{R}^n$ , then  $S_{\sharp} \in \text{Hom}(\mathbf{R}^n, \mathbf{R}^n)$  shall denote the orthogonal projection onto  $S$ . In particular if  $p \in \mathbf{O}^*(n, m)$  is such that  $\text{im } p^* = S$ , then  $S_{\sharp} = p^* \circ p$ .

Whenever  $\mu$  is a (Radon) measure over some set  $U \subseteq \mathbf{R}^n$  we sometimes use the same symbol  $\mu$  to denote the (not necessarily Radon) measure  $j_{\sharp}\mu$  over  $\mathbf{R}^n$ , where  $j : U \rightarrow \mathbf{R}^n$  is the inclusion map. Nonetheless, the support of  $\mu$  is always a subset of  $U$ , i.e.,  $\text{spt } \mu \subseteq U$ .

### 3 Statement of the main result

**3.1 Definition.** Let  $U \subseteq \mathbf{R}^n$  be open. We say that  $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$  is a *basic deformation in  $U$*  if  $f$  is of class  $\mathcal{C}^1$  and there exist a bounded convex open set  $V \subseteq U$  such that

$$f(x) = x \quad \text{for } x \in \mathbf{R}^n \sim V \quad \text{and} \quad f[V] \subseteq V.$$

If  $f \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$  is a composition of a finite number of basic deformations, then we say that  $f$  is an *admissible deformation in  $U$* . The set of all such deformations shall be denoted  $\mathfrak{D}(U)$ .

**3.2 Remark.** In most cases the bounded convex set  $V$  shall be a cube or a ball.

**3.3 Definition.** Whenever  $K \subseteq \mathbf{R}^n$  is compact and  $A, B \subseteq \mathbf{R}^n$ , we define  $d_{\mathcal{H}, K}(A, B)$  by

$$d_{\mathcal{H}, K}(A, B) = \max\{\sup\{\text{dist}(x, A) : x \in K \cap B\}, \sup\{\text{dist}(x, B) : x \in K \cap A\}\}.$$

**3.4 Definition.** Let  $U \subseteq \mathbf{R}^n$  be an open set. We say that  $\mathcal{C}$  is a *good class in  $U$*  if

- (a)  $\mathcal{C} \neq \emptyset$ ;
- (b) each  $S \in \mathcal{C}$  is a closed subset of  $\mathbf{R}^n$ ;
- (c) if  $S \in \mathcal{C}$  and  $f \in \mathfrak{D}(U)$ , then  $f[S] \in \mathcal{C}$ ;
- (d) if  $S_i \in \mathcal{C}$  for  $i \in \mathscr{P}$ , and  $S \subseteq \mathbf{R}^n$ , and  $\lim_{i \rightarrow \infty} d_{\mathcal{H}, K}(S_i \cap U, S \cap U) = 0$  for all compact sets  $K \subseteq U$ , then  $S \in \mathcal{C}$ .

**3.5 Remark.** One example of a good class is given in 12.4. We expect that the methods presented in this article could work also if we assumed that admissible deformations are uniform limits of diffeomorphisms (so called *monotone maps*) as in [DPDRG16, Definition 1.1]. In such case, the class denoted  $\mathcal{F}(H, \mathcal{C})$  defined in [DPDRG16, Definition 1.4] would also be good. However, we had trouble checking that all the deformations we use are monotone. In particular, the deformations constructed in 9.1 are clearly not monotone and there is no easy way to fix that. We anticipate that one could modify the deformation theorem 7.13 to handle the situation from 9.1 and provide an appropriate monotone map but, given the overall complexity of the already presented material, we chose not to do that.

**3.6 Definition** (cf. [Alm68, 1.2]). A function  $F : \mathbf{R}^n \times \mathbf{G}(n, m) \rightarrow [0, \infty)$  of class  $\mathcal{C}^k$  for some non-negative integer  $k \in \mathscr{P}$  is called a  $\mathcal{C}^k$  *integrand*.

If additionally  $\inf \text{im } F / \sup \text{im } F \in (0, \infty)$ , then we say that  $F$  is *bounded*.

**3.7 Definition** (cf. [Alm68, 3.1]). If  $\varphi \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$  and  $F$  is an integrand, then the *pull-back* integrand  $\varphi^\# F$  is given by

$$\varphi^\# F(x, T) = \begin{cases} F(\varphi(x), D\varphi(x)[T]) \|\bigwedge_m D\varphi(x) \circ T_\sharp\| & \text{if } \dim D\varphi(x)[T] = m \\ 0 & \text{if } \dim D\varphi(x)[T] < m. \end{cases}$$

If  $\varphi$  is a diffeomorphism, then the *push-forward* integrand is given by  $\varphi_\# F = (\varphi^{-1})^\# F$ .

**3.8 Definition** (cf. [Alm68, 1.2]). If  $F$  is a  $\mathcal{C}^k$  integrand and  $x \in \mathbf{R}^n$ , then we define another  $\mathcal{C}^k$  integrand  $F^x$  by the formula

$$F^x(y, S) = F(x, S) \quad \text{for } y \in \mathbf{R}^n \text{ and } S \in \mathbf{G}(n, m).$$

**3.9 Definition** (cf. [All72, 3.5]). Assume  $S \subseteq \mathbf{R}^n$  is such that  $\mathcal{H}^m(S \cap K) < \infty$  for any compact set  $K \subseteq \mathbf{R}^n$ . We define  $\mathbf{v}(S) \in \mathbf{V}_m(\mathbf{R}^n)$  in the following way: first decompose  $S$  into a sum  $S_u \cup S_r$ , where  $S_u$  is purely  $(\mathcal{H}^m, m)$  unrectifiable and  $S_r$  is countably  $(\mathcal{H}^m, m)$  rectifiable and then set

$$\begin{aligned} \mathbf{v}(S)(\alpha) &= \int_{S_r} \alpha(x, \text{Tan}^m(\mathcal{H}^m \llcorner S_r, x)) \, d\mathcal{H}^m(x) \\ &\quad + \int_{S_u} \int \alpha(x, T) \, d\gamma_{n,m}(T) \, d\mathcal{H}^m(x) \quad \text{for } \alpha \in \mathcal{K}(\mathbf{R}^n \times \mathbf{G}(n, m)), \end{aligned}$$

where  $\gamma_{n,m}$  denotes the canonical probabilistic measure on  $\mathbf{G}(n, m)$  invariant under the action of the orthogonal group  $\mathbf{O}(n)$ ; see [Fed69, 2.7.16(6)].

**3.10 Definition.** If  $F$  is a  $\mathcal{C}^k$  integrand, we define the functional  $\Phi_F : \mathbf{V}_m(\mathbf{R}^n) \rightarrow [0, \infty]$  by the formula

$$\Phi_F(V) = \int F(x, S) \, dV(x, S).$$

**3.11 Remark.** If  $\text{spt } \|V\|$  is compact we have  $\Phi_F(V) = V(\gamma F)$ , whenever  $\gamma \in \mathcal{D}(\mathbf{R}^n, \mathbf{R})$  is such that  $\text{spt } \|V\| \subseteq \text{Int } \gamma^{-1}\{1\}$ .

**3.12 Definition.** If  $S \subseteq \mathbf{R}^n$  satisfies  $\mathcal{H}^m(S \cap K) < \infty$  for all compact sets  $K \subseteq \mathbf{R}^n$ , then we set

$$\Phi_F(S) = \Phi_F(\mathbf{v}(S)).$$

Otherwise we set  $\Phi_F(S) = \infty$ .

We shall use the following notion of ellipticity introduced by Almgren. It can be understood as a geometric version of quasi-convexity; cf. [Mor66].

**3.13 Definition** (cf. [Fed69, 5.1.2]). A  $\mathcal{C}^k$  integrand  $F$  is called *elliptic* if there exists a continuous function  $c : \mathbf{R}^n \rightarrow (0, \infty)$  such that for all  $T \in \mathbf{G}(n, m)$ , and  $p \in \mathbf{R}^n$ , and  $r \in (0, \infty)$  we have

$$\Phi_{F^x}(S) - \Phi_{F^x}(D) \geq c(x)(\mathcal{H}^m(S) - \mathcal{H}^m(D))$$

whenever

- (a)  $D = \{z \in \mathbf{R}^n : |z - p| \leq r, z - p \in T\}$  is an  $m$  dimensional disc in  $\mathbf{R}^n$  parallel to  $T$  centred at  $p$  of radius  $r$ ;

- (b)  $S$  is a closed subset of  $\mathbf{R}^n$  such that setting  $A = \{z \in \mathbf{R}^n : |z - p| = r, z - p \in T\}$  we have  $A \subseteq S$  and  $S \cap \mathbf{B}(p, r)$  cannot be deformed onto  $A$  (i.e.  $f[S \cap \mathbf{B}(p, r)] \not\subseteq A$ ) by any deformation  $f \in \mathcal{D}(\mathbf{U}(p, r))$ .

**3.14 Remark.** Note the following observations.

- The notion of ellipticity depends on the choice of admissible maps. The more maps in  $\mathcal{D}(\mathbf{U}(p, r))$  the less surfaces  $S$  that satisfy 3.13(b); hence, more elliptic integrands.
- The *area integrand*  $F \equiv 1$  is elliptic.
- If  $F$  is an integrand and  $\varphi \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$  is a diffeomorphism, then  $F$  is elliptic if and only if  $\varphi_{\#}F$  is elliptic; cf. [Alm68, 4.3].
- Any convex combination of elliptic integrands is elliptic.

**3.15 Remark.** It is not clear whether strict convexity of  $F$  in the second variable is enough to ensure ellipticity of  $F$  as is the case in the context of currents; see [Fed69, 5.1.2]. De Lellis, De Rosa, and Ghiraldin were able to prove their existence result assuming  $F$  is uniformly convex in the second variable and of class  $\mathcal{C}^2$ ; see [DDG16a, Definition 2.4]. De Philippis, De Rosa, and Ghiraldin defined so called *atomic condition* for the integrand; see [DDG16b, Definition 1]. In co-dimension one this condition is equivalent to strict convexity of  $F$  in the second variable; see [DDG16b, Theorem 1.3]. Moreover, varifolds whose first variation with respect to  $F$ , satisfying the atomic condition, induces a Radon measure are rectifiable; see [DDG16b, Theorem 1.2].

Our main theorem reads.

**3.16 Theorem.** *Let  $U \subset \mathbf{R}^n$  be an open set,  $\mathcal{C}$  be a good class in  $U$ , and  $F$  be a bounded elliptic  $\mathcal{C}^0$  integrand. Set  $\mu = \inf\{\Phi_F(T \cap U) : T \in \mathcal{C}\}$ .*

*If  $\mu \in (0, \infty)$ , then there exist  $S \in \mathcal{C}$  and a sequence  $\{S_i \in \mathcal{C} : i \in \mathcal{P}\}$  such that*

*(a)  $S \cap U$  is  $(\mathcal{H}^m, m)$  rectifiable. In particular  $\mathcal{H}^m(S \cap U) < \infty$ .*

*(b)  $\lim_{i \rightarrow \infty} \Phi_F(S_i \cap U) = \Phi_F(S \cap U) = \mu$ .*

*(c)  $\lim_{i \rightarrow \infty} \mathbf{v}(S_i \cap U) = \mathbf{v}(S \cap U)$  in  $\mathbf{V}_m(U)$ .*

*(d)  $\lim_{i \rightarrow \infty} d_{\mathcal{H}, K}(S_i \cap U, S \cap U) = 0$  for any compact set  $K \subseteq U$ .*

*Furthermore, if  $\mathbf{R}^n \sim U$  is compact and there exists a  $\Phi_F$ -minimising sequence in  $\mathcal{C}$  consisting only of compact sets (but not necessarily uniformly bounded), then*

$$\text{diam}(\text{spt} \|V\|) < \infty \quad \text{and} \quad \sup\{\text{diam}(S_i \cap U) : i \in \mathcal{P}\} < \infty.$$

## 4 Unrectifiable sets under submersions

Assume  $K \subseteq \mathbf{R}^n$  is purely  $(\mathcal{H}^m, m)$  unrectifiable with  $\mathcal{H}^m(K) < \infty$  and  $f \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$  is such that  $Df(x)$  is of rank at most  $m$  for  $x \in \mathbf{R}^n$ . We construct an arbitrarily small  $\mathcal{C}^1$  perturbation  $\tilde{f}$  of  $f$  such that  $\mathcal{H}^m(\tilde{f}[K])$  becomes very small. Additionally, we ensure that  $\tilde{f}$  is of the form  $\tilde{f} = f \circ \rho$ , where  $\rho$  is a diffeomorphism of  $\mathbf{R}^n$  such that  $\text{Lip}(\rho - \text{id}_{\mathbf{R}^n})$  is very small. This provides a useful feature of  $\tilde{f}$ , namely that  $\text{im } \tilde{f} \subseteq \text{im } f$ .

A similar result was proven recently by Pugh [Pug16]. It could be possible to obtain  $\mathcal{H}^m(\tilde{f}[K]) = 0$  as was shown by Gałęski [Gał16] but for our purposes it suffices only to

make the measure small. Also the map constructed in [Gal16] kills only the measure of the part of  $K$  on which  $\dim \operatorname{im} Df(x) = m$  and we need to take care also of the part where the rank of  $Df$  is strictly less than  $m$ . Finally, we should mention that Almgren alluded that such result should hold already in [Alm68, 2.9(b1)].

In the next preparatory lemma we construct a smooth map  $M : \mathbf{R} \rightarrow \mathbf{O}(n)$  which continuously rotates a given  $m$ -plane  $S$  onto another given  $m$ -plane  $T$ . We also derive estimates on  $M'$  as well as on  $\|M(\cdot) - \operatorname{id}_{\mathbf{R}^n}\|$  in terms of  $\|S_{\mathfrak{h}} - T_{\mathfrak{h}}\|$ .

**4.1 Lemma.** *Let  $n$  and  $m$  be positive integers such that  $0 < m \leq n$ . There exists  $\Gamma \in (0, \infty)$  such that for all  $S, T \in \mathbf{G}(n, m)$  there exists  $M : \mathbf{R} \rightarrow \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$  of class  $\mathcal{C}^\infty$  satisfying*

$$(1) \quad \begin{aligned} &M(0) = \operatorname{id}_{\mathbf{R}^n}, \quad M(1)[S] = T, \quad \forall \tau \in \mathbf{R} \quad M(\tau) \in \mathbf{O}(n), \\ &\forall \tau \in \mathbf{R} \quad \|M(\tau) - \operatorname{id}_{\mathbf{R}^n}\| \leq \Gamma |\tau| \|S_{\mathfrak{h}} - T_{\mathfrak{h}}\| \quad \text{and} \quad \|M'(\tau)\| \leq \Gamma \|S_{\mathfrak{h}} - T_{\mathfrak{h}}\|. \end{aligned}$$

*Proof.* We shall construct the map  $M$  similarly as in [All72, 8.9(3)]. First set

$$\begin{aligned} A &= S \cap T, \quad B = S^\perp \cap T^\perp = (S + T)^\perp, \\ C &= (S^\perp \cap T) \oplus (T^\perp \cap S), \quad D = (S + T) \cap A^\perp \cap C^\perp. \end{aligned}$$

Observe that  $A, B, C, D$  are pairwise orthogonal and sum up to the whole of  $\mathbf{R}^n$ , i.e.,

$$\forall X, Y \in \{A, B, C, D\} \quad X = Y \text{ or } X \perp Y, \quad \mathbf{R}^n = A \oplus B \oplus C \oplus D.$$

Note also that there exist natural numbers  $k$  and  $l$  such that

$$\begin{aligned} k &= \dim(S \cap D) = \dim(T \cap D), \quad \dim(D) = \dim(S \cap D) + \dim(T \cap D) = 2k, \\ l &= \dim(S^\perp \cap T) = \dim(T^\perp \cap S), \quad \dim(C) = 2l. \end{aligned}$$

For our convenience we set  $S_0 = S$  and  $T_0 = T$ . If  $k > 0$ , then we shall construct inductively

- subspaces  $S \supseteq S_1 \supseteq S_2 \supseteq \dots \supseteq S_k$  and  $T \supseteq T_1 \supseteq T_2 \supseteq \dots \supseteq T_k$  and  $V_1, \dots, V_k \subseteq S + T$ ,
- and vectors  $s_1 \in S_1, \dots, s_k \in S_k$  and  $t_1 \in T_1, \dots, t_k \in T_k$ .

To start the construction we set  $S_1 = S \cap D$  and  $T_1 = T \cap D$ . Then we use [All72, 8.9(3)] to find  $s_1 \in S_1$  so that  $|s_1| = 1$  and  $|(T_1^\perp)_{\mathfrak{h}} s_1| = \|S_{1\mathfrak{h}} - T_{1\mathfrak{h}}\|$ . Note that  $\|S_{1\mathfrak{h}} - T_{1\mathfrak{h}}\| < 1$  because the spaces  $S_1$  and  $T_1$  are orthogonal to  $C$ . Hence, we may define  $t_1 = T_{1\mathfrak{h}} s_1 |T_{1\mathfrak{h}} s_1|^{-1}$  and  $V_1 = \operatorname{span}\{s_1, t_1\}$ . Assuming we have constructed  $S_1, \dots, S_i$  and  $T_1, \dots, T_i$  and  $s_1, \dots, s_i$  and  $t_1, \dots, t_i$  for some  $i \in \{1, \dots, k-1\}$  we proceed by requiring

$$\begin{aligned} S_{i+1} &= S_i \cap V_i^\perp, \quad T_{i+1} = T_i \cap V_i^\perp, \quad s_{i+1} \in S_{i+1}, \quad |s_{i+1}| = 1, \\ |T_{i+1\mathfrak{h}}^\perp s_{i+1}| &= \|S_{i+1\mathfrak{h}} - T_{i+1\mathfrak{h}}\|, \quad t_{i+1} = \frac{T_{i+1\mathfrak{h}} s_{i+1}}{|T_{i+1\mathfrak{h}} s_{i+1}|}, \quad V_{i+1} = \operatorname{span}\{s_{i+1}, t_{i+1}\}. \end{aligned}$$

Observe that

$$\forall i \in \{0, 1, \dots, k-1\} \quad \forall s \in S_{i+1} \subseteq S_i \quad (T_i^\perp)_{\mathfrak{h}} s = T_{i+1\mathfrak{h}}^\perp s;$$

thus,

$$\begin{aligned} \|S_{i+1\mathfrak{h}} - T_{i+1\mathfrak{h}}\| &= \sup\{|T_{i+1\mathfrak{h}} s| : s \in S_{i+1}, |s| = 1\} \\ &= \sup\{|T_{i\mathfrak{h}} s| : s \in S_{i+1}, |s| = 1\} \leq \sup\{|T_{i\mathfrak{h}} s| : s \in S_i, |s| = 1\}. \end{aligned}$$



Therefore,

$$\forall i \in \{0, 1, \dots, k\} \quad \|S_{i\mathfrak{h}} - T_{i\mathfrak{h}}\| \leq \|S_{\mathfrak{h}} - T_{\mathfrak{h}}\|.$$

Clearly  $(s_1, \dots, s_k)$  and  $(t_1, \dots, t_k)$  are orthonormal bases of  $S \cap D$  and  $T \cap D$  respectively. Next, we choose arbitrary orthonormal bases  $(s_{k+1}, \dots, s_{k+l})$  of  $S \cap T^\perp$  and  $(t_{k+1}, \dots, t_{k+l})$  of  $T \cap S^\perp$  and  $(e_1, \dots, e_{n-2(k+l)})$  of  $A \oplus B$ . We also define

$$\alpha_i = \arccos(s_i \bullet t_i) \quad \text{for } i \in \{1, \dots, k+l\}$$

and note that  $0 < \alpha_i \leq \pi/2$  by construction. Now we are in position to define  $M$ . It shall be the identity on  $A \oplus B$  and on each  $V_i = \text{span}\{s_i, t_i\}$  it will be the rotation sending  $s_i$  to  $t_i$  for  $i = 1, 2, \dots, k+l$ . More precisely we set

$$\hat{s}_i = \frac{t_i - (t_i \bullet s_i)s_i}{|t_i - (t_i \bullet s_i)s_i|} \quad \text{for } i \in \{1, \dots, k\}, \quad \hat{s}_i = t_i \quad \text{for } i \in \{k+1, \dots, k+l\},$$

and define for  $\tau \in \mathbf{R}$

$$\begin{aligned} M(\tau)s_i &= \cos(\tau\alpha_i)s_i + \sin(\tau\alpha_i)\hat{s}_i \quad \text{for } i = 1, \dots, k+l, \\ M(\tau)\hat{s}_i &= -\sin(\tau\alpha_i)s_i + \cos(\tau\alpha_i)\hat{s}_i \quad \text{for } i = 1, \dots, k+l, \\ M(\tau)e_i &= e_i \quad \text{for } i = 1, \dots, n-2(k+l) \end{aligned}$$

Since  $\{s_1, \dots, s_{k+l}, \hat{s}_1, \dots, \hat{s}_{k+l}, e_1, \dots, e_{n-2(k+l)}\}$  is an orthonormal basis of  $\mathbf{R}^n$  we see that  $M(\tau) \in \mathbf{O}(n)$  for each  $\tau \in \mathbf{R}$ . It is also immediate that  $M(0) = \text{id}_{\mathbf{R}^n}$  and  $M(1)[S] = T$ .

To prove (1) we first estimate  $\alpha_i$ . Recall that  $1 - \cos x = 2\sin^2(x/2)$  for  $x \in \mathbf{R}$  and  $|x| \leq 2|\sin x|$  whenever  $|x| \leq \pi/2$ ; hence, for  $i = 1, \dots, k+l$

$$\begin{aligned} \alpha_i &\leq 4\sin(\alpha_i/2) = 2\sqrt{2}(1 - \cos(\alpha_i))^{1/2} = 2\sqrt{2}(1 - s_i \bullet t_i)^{1/2} \\ &= 2\sqrt{2}(1 - (1 - \|T_{i\mathfrak{h}}^\perp s_i\|^2)^{1/2})^{1/2} \leq 2\sqrt{2}(1 - (1 - \|T_{\mathfrak{h}} - S_{\mathfrak{h}}\|^2)^{1/2})^{1/2}. \end{aligned}$$

If  $\|T_{\mathfrak{h}} - S_{\mathfrak{h}}\| < 1/2$ , then we use standard estimates  $\exp(x) \geq 1+x$  and  $\log(1+x) \geq x/(1+x)$  valid for  $x > -1$  to derive

$$(2) \quad \alpha_i \leq 2\sqrt{2} \frac{\|T_{\mathfrak{h}} - S_{\mathfrak{h}}\|}{(1 - \|T_{\mathfrak{h}} - S_{\mathfrak{h}}\|^2)^{1/2}} \leq 8\|T_{\mathfrak{h}} - S_{\mathfrak{h}}\|.$$

If  $\|T_{\mathfrak{h}} - S_{\mathfrak{h}}\| \geq 1/2$ , we have  $(1 - (1 - \|T_{\mathfrak{h}} - S_{\mathfrak{h}}\|^2)^{1/2})^{1/2} \leq 1 \leq 2\|T_{\mathfrak{h}} - S_{\mathfrak{h}}\|$  so (2) holds also in this case.

Using  $|\sin x| \leq |x|$  for  $x \in \mathbf{R}$  and (2) we obtain for  $i = 1, 2, \dots, k+l$  and  $\tau \in \mathbf{R}$

$$|s_i - M(\tau)s_i| = 2|\sin(\tau\alpha_i/2)| \leq |\tau|\alpha_i \leq 8|\tau|\|T_{\mathfrak{h}} - S_{\mathfrak{h}}\|.$$

Thus, whenever  $v \in \mathbf{R}^n$  and  $|v| = 1$ ,

$$|v - M(\tau)v|^2 = \sum_{i=1}^{k+l} |V_{i\mathfrak{h}}v - M(\tau)(V_{i\mathfrak{h}}v)|^2 = \sum_{i=1}^{k+l} |V_{i\mathfrak{h}}v|^2 |s_i - M(\tau)s_i|^2 \leq (8|\tau|\|T_{\mathfrak{h}} - S_{\mathfrak{h}}\|)^2,$$

which proves the first part of (1). By direct computation we obtain  $|M'(\tau)s_i| = |M'(\tau)\hat{s}_i| = \alpha_i$  for  $\tau \in \mathbf{R}$  and  $i = 1, 2, \dots, k+l$ . Therefore, employing (2),

$$\|M'(\tau)\| = \max\{\alpha_i : i = 1, 2, \dots, k+l\} \leq 8\|T_{\mathfrak{h}} - S_{\mathfrak{h}}\|. \quad \square$$

The following technical lemma is a localised and reparameterised version of 4.1. Roughly speaking, we construct a diffeomorphism  $\rho$  of  $\mathbf{R}^n$  which acts as a rotation inside a given ball and is the identity outside some neighbourhood of that ball. To be able to utilise 4.2 in 4.3 we need to perform the rotation in different coordinates, which is accomplished by passing through a diffeomorphism  $\varphi$ . We use estimates from 4.1 to bound  $\text{Lip}(\rho - \text{id}_{\mathbf{R}^n})$ .

**4.2 Lemma.** *Assume*

$$\begin{aligned}
(3) \quad & k \in \mathcal{P}, \quad U \subseteq \mathbf{R}^n \text{ is open}, \quad q \in \mathbf{O}^*(n, m), \quad T = \text{im } q^*, \quad S \in \mathbf{G}(n, m), \\
& a \in U, \quad \tilde{r}, r \in \mathbf{R}, \quad 0 < r < \text{dist}(a, \mathbf{R}^n \setminus U), \quad 0 < \tilde{r} < r, \\
& \varphi \in \mathcal{C}^k(U, \mathbf{R}^n) \text{ is a diffeomorphism onto its image}, \\
& L = \sup\{\max\{\|D\varphi(x)\|, \|D\varphi(x)^{-1}\|\} : x \in \mathbf{B}(a, r)\}, \\
& \omega(s) = \sup\{\|D\varphi(x) - D\varphi(y)\| : x, y \in \mathbf{B}(a, r), |x - y| \leq s\} \quad \text{for } s \in \mathbf{R}, s \geq 0, \\
& \Gamma = \Gamma(L, r, \tilde{r}) = (2L^2r/(r - \tilde{r}) + 1)\Gamma_{4.1}, \quad \Gamma\|S_{\mathfrak{h}} - T_{\mathfrak{h}}\| < 1.
\end{aligned}$$

*Then there exist a diffeomorphism  $\rho \in \mathcal{C}^k(\mathbf{R}^n, \mathbf{R}^n)$  and  $p \in \mathbf{O}^*(n, m)$  such that*

$$\begin{aligned}
(4) \quad & \rho(x) = x \quad \text{for } x \in \mathbf{R}^n \setminus \mathbf{U}(a, r), \quad \text{im } p^* = S, \\
(5) \quad & q \circ \varphi \circ \rho(x) = p(\varphi(x) - \varphi(a)) + q(\varphi(a)) \quad \text{for } x \in \mathbf{B}(a, \tilde{r}), \\
(6) \quad & \sup\{\|D\rho(x) - \text{id}_{\mathbf{R}^n}\| : x \in \mathbf{R}^n\} \leq 2L\omega(Lr\Gamma\|S_{\mathfrak{h}} - T_{\mathfrak{h}}\|) + L^2\Gamma\|S_{\mathfrak{h}} - T_{\mathfrak{h}}\|.
\end{aligned}$$

*Proof.* Employ 4.1 to obtain a smooth map  $M : \mathbf{R} \rightarrow \text{Hom}(\mathbf{R}^n, \mathbf{R}^n)$  such that  $M(1)[S] = T$  and  $M(\tau) \in \mathbf{O}(n)$  for each  $\tau \in \mathbf{R}$ . Let  $\zeta : \mathbf{R} \rightarrow \mathbf{R}$  be of class  $\mathcal{C}^\infty$  and satisfy  $\zeta(t) = 0$  for  $t \leq 0$ , and  $\zeta(t) = 1$  for  $t \geq 1$ , and  $0 \leq \zeta'(t) \leq 2$  for  $t \in \mathbf{R}$ , and  $0 \in \text{Int } \zeta^{-1}\{0\}$ , and  $1 \in \text{Int } \zeta^{-1}\{1\}$ . Define  $\pi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ , and  $\eta : \mathbf{R}^n \rightarrow \mathbf{R}^n$ , and  $p \in \text{Hom}(\mathbf{R}^n, \mathbf{R}^m)$  by requiring

$$\begin{aligned}
& \eta(x) = (r - |\varphi^{-1}(x) - a|)/(r - \tilde{r}) \quad \text{if } x \in \varphi[\mathbf{B}(a, r)], \\
& \eta(x) = 0 \quad \text{if } x \in \mathbf{R}^n \setminus \varphi[\mathbf{B}(a, r)], \quad p = q \circ M(1), \\
& \pi(x) = M \circ \zeta \circ \eta(x)(x - \varphi(a)) + \varphi(a) \quad \text{for } x \in \mathbf{R}^n.
\end{aligned}$$

Note that  $\eta$  is Lipschitz continuous and  $\pi$  is of class  $\mathcal{C}^k$  because  $0 \in \text{Int } \zeta^{-1}\{0\}$  and  $1 \in \text{Int } \zeta^{-1}\{1\}$ . Moreover,  $p \in \mathbf{O}^*(n, m)$  and

$$\begin{aligned}
(7) \quad & \text{im } p^* = M(1)^* \circ q^*[\mathbf{R}^m] = M(1)^{-1}[T] = S, \\
(8) \quad & \pi(x) = x \quad \text{whenever } x \in \mathbf{R}^n \setminus \varphi[\mathbf{U}(a, r)], \\
(9) \quad & q \circ \pi(x) = p(x - \varphi(a)) + q(\varphi(a)) \quad \text{for } x \in \varphi[\mathbf{B}(a, \tilde{r})].
\end{aligned}$$

Hence, we can set

$$\rho = \varphi^{-1} \circ \pi \circ \varphi.$$

Clearly (7), (8), (9) imply (4) and (5) and we only need to check (6). Recalling (1) and (3) and  $\text{Lip}(\varphi|_{\mathbf{B}(a, r)}) \leq L$  we estimate for  $x \in \mathbf{B}(a, r)$

$$\begin{aligned}
(10) \quad & \text{Lip}(\eta) \leq \text{Lip}((\varphi|_{\mathbf{B}(a, r)})^{-1})/(r - \tilde{r}) \leq L/(r - \tilde{r}), \\
& \|D(\pi - \text{id}_{\mathbf{R}^n})(\varphi(x))\| \leq \text{Lip}(\zeta) \text{Lip}(\eta) \|M'(\zeta \circ \eta(\varphi(x)))\| Lr + \|M(\zeta \circ \eta(\varphi(x))) - \text{id}_{\mathbf{R}^n}\| \\
& \leq (2L^2r/(r - \tilde{r}) + 1)\Gamma_{4.1}\|S_{\mathfrak{h}} - T_{\mathfrak{h}}\| = \Gamma\|S_{\mathfrak{h}} - T_{\mathfrak{h}}\| < 1.
\end{aligned}$$

In particular, using (8) we conclude that  $\text{Lip}(\pi - \text{id}_{\mathbf{R}^n}) < 1$ , so  $\pi$  and  $\rho$  are diffeomorphisms. Employing (10) we see also that if  $x \in \mathbf{B}(a, r)$ , then

$$(11) \quad |\pi(\varphi(x)) - \varphi(x)| = |(\pi - \text{id}_{\mathbf{R}^n})(\varphi(x)) - (\pi - \text{id}_{\mathbf{R}^n})(\varphi(y))| \leq Lr\Gamma\|S_{\natural} - T_{\natural}\|,$$

where  $y \in U \sim \mathbf{U}(a, r)$  is any point such that  $|x - y| = \text{dist}(x, \mathbf{R}^n \sim \mathbf{B}(a, r)) \leq r$ . Utilising (10) and (11) we see that  $\text{Lip}(\pi) \leq 2$  and for  $x \in \mathbf{B}(a, r)$

$$\begin{aligned} \|\text{D}\rho(x) - \text{id}_{\mathbf{R}^n}\| &\leq \|(\text{D}\varphi^{-1}(\pi \circ \varphi(x)) - \text{D}\varphi^{-1}(\varphi(x))) \circ \text{D}\pi(\varphi(x))\| \|\text{D}\varphi(x)\| \\ &\quad + \|\text{D}\varphi^{-1}(\varphi(x)) \circ (\text{D}\pi(\varphi(x)) - \text{id}_{\mathbf{R}^n})\| \|\text{D}\varphi(x)\| \\ &\leq 2L\omega(|\pi(\varphi(x)) - \varphi(x)|) + L^2\|\text{D}(\pi - \text{id}_{\mathbf{R}^n})(\varphi(x))\| \\ &\leq 2L\omega(Lr\Gamma\|S_{\natural} - T_{\natural}\|) + L^2\Gamma\|S_{\natural} - T_{\natural}\|. \quad \square \end{aligned}$$

In the next lemma given a purely  $(\mathcal{H}^m, m)$  unrectifiable set  $K$  with  $\mathcal{H}^m(K) < \infty$  and a map  $f \in \mathcal{C}^k(\mathbf{R}^n, \mathbf{R}^n)$  such that  $\text{D}f(x)$  is of rank at most  $m$  for  $x \in \mathbf{R}^n$  we employ the constant rank theorem together with a Vitali covering theorem to get a family of balls in each of which we apply 4.2 and the Besicovitch–Federer projection theorem to construct a diffeomorphism  $\rho$  of  $\mathbf{R}^n$  such that  $f \circ \rho[K]$  has significantly less  $\mathcal{H}^m$  measure than  $K$  itself. Since, in general,  $f$  may map the set where  $\text{D}f(x)$  has rank strictly less than  $m$  into a set of positive  $\mathcal{H}^m$  measure we need to additionally assume that this does not happen or assume  $k \geq n - m + 1$  and employ the Morse–Sard theorem; see 4.4.

**4.3 Lemma.** *Let  $K \subseteq \mathbf{R}^n$  be purely  $(\mathcal{H}^m, m)$  unrectifiable with  $\mathcal{H}^m(K) < \infty$ . Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be of class  $\mathcal{C}^k$  with  $k \geq 1$ . Suppose there exists an open set  $U \subseteq \mathbf{R}^n$  such that*

$$(12) \quad \begin{aligned} &K \subseteq U \quad \text{and} \quad \dim \text{im } \text{D}f(x) \leq m \text{ for all } x \in U \\ &\text{and} \quad \mathcal{H}^m(f[\{x \in K : \dim \text{im } \text{D}f(x) < m\}]) = 0. \end{aligned}$$

*Then for any  $\varepsilon \in (0, \infty)$  there exists a diffeomorphism  $\rho_\varepsilon : \mathbf{R}^n \rightarrow \mathbf{R}^n$  of class  $\mathcal{C}^k$  such that*

$$\begin{aligned} \mathcal{H}^m(f \circ \rho_\varepsilon[K]) &\leq \varepsilon \mathcal{H}^m(K), \quad \rho_\varepsilon(x) = x \quad \text{for } x \in \mathbf{R}^n \sim U, \\ |x - \rho_\varepsilon(x)| &\leq \varepsilon \quad \text{and} \quad \|\text{id}_{\mathbf{R}^n} - \text{D}\rho_\varepsilon(x)\| \leq \varepsilon \quad \text{for } x \in \mathbf{R}^n. \end{aligned}$$

*Proof.* Let  $\varepsilon \in (0, \infty)$  and let  $q : \mathbf{R}^m \times \mathbf{R}^{n-m} \rightarrow \mathbf{R}^n$  be given by  $q(x, y) = x$ . Set

$$A = \{x \in U : \dim \text{im } \text{D}f(x) = m\}.$$

Since  $\dim \text{im } \text{D}f(x) \leq m$  for all  $x \in U$  we see that  $A = \{x \in U : \bigwedge_m \text{D}f(x) \neq 0\}$  is open. Hence, for every  $a \in A$  the constant rank theorem [Fed69, 3.1.18] ensures the existence of open sets  $U_a \subseteq U$ ,  $V_a \subseteq \mathbf{R}^n$ , maps  $\varphi_a : U_a \rightarrow \mathbf{R}^n$ ,  $\psi_a : V_a \rightarrow \mathbf{R}^n$  which are diffeomorphisms onto their respective images, and orthogonal projections  $p_a \in \mathbf{O}^*(n, m)$  such that

$$a \in U_a, \quad f(a) \in V_a, \quad f|_{U_a} = \psi_a^{-1} \circ p_a^* \circ q \circ \varphi_a.$$

Applying the Vitali covering theorem (see [Fed69, 2.8.16, 2.8.18] or alternatively [Mat95, 2.8]) to the measure  $\mathcal{H}^m \llcorner K$  and the family of all the closed balls  $\mathbf{B}(a, r)$  satisfying

$$(13) \quad a \in K \cap A, \quad 0 < r < \min\{1, \varepsilon\}, \quad \mathbf{B}(a, r) \subseteq U_a,$$

$$(14) \quad \lim_{s \uparrow r} \mathcal{H}^m(K \cap \mathbf{B}(a, r) \sim \mathbf{B}(a, s)) = 0$$

we obtain a countable disjointed collection  $\mathcal{B}$  of closed balls having the properties (13), (14), and additionally

$$(15) \quad \mathcal{H}^m((K \cap A) \sim \bigcup \mathcal{B}) = 0.$$

Whenever  $\mathbf{B}(a, r) \in \mathcal{B}$  we set

$$L_a = \max \{ \text{Lip}(\varphi_a|_{\mathbf{B}(a, r)}), \text{Lip}((\varphi_a|_{\mathbf{B}(a, r)})^{-1}), \text{Lip}(f|_{\mathbf{B}(a, r)}) \}.$$

Set  $I = \{a \in \mathbf{R}^n : \mathbf{B}(a, r) \in \mathcal{B} \text{ for some } r \in \mathbf{R}\}$  and  $T = \text{im}(q^*) = \mathbf{R}^m \times \{0\} \in \mathbf{G}(n, m)$ . Whenever  $a \in I$  define  $r_a \in \mathbf{R}$  to be the unique number such that  $\mathbf{B}(a, r_a) \in \mathcal{B}$ . Suppose  $a \in I$ . Since  $\varphi_a$  is a diffeomorphism onto its image, we see that  $\varphi_a[K \cap \mathbf{B}(a, r_a)]$  is purely  $(\mathcal{H}^m, m)$  unrectifiable. Hence, the Besicovitch–Federer projection theorem (see [Fed69, 3.3.15] or alternatively [Mat95, 18.1]) allows us to find a sequence of  $m$ -planes  $S_{a, i} \in \mathbf{G}(n, m)$  such that  $\|S_{a, i} - T_{\mathfrak{h}}\| \rightarrow 0$  as  $i \rightarrow \infty$  and

$$(16) \quad \mathcal{H}^m(S_{a, i_{\mathfrak{h}}}[\varphi_a[K \cap \mathbf{B}(a, r_a)]]) = 0 \quad \text{for all } i \in \mathcal{P}.$$

Using (14) we find  $\tilde{r}_a \in \mathbf{R}$  such that  $0 < \tilde{r}_a < r_a$  and

$$(17) \quad \mathcal{H}^m(K \cap \mathbf{B}(a, r_a) \sim \mathbf{B}(a, \tilde{r}_a)) < (2L_a)^{-m} \varepsilon \mathcal{H}^m(K \cap \mathbf{B}(a, r_a)).$$

Set  $\Delta = \Gamma_{4.2}(L_a, r_a, \tilde{r}_a)$  and

$$\omega_a(s) = \sup \{ \|D\varphi_a(x) - D\varphi_a(y)\| : x, y \in \mathbf{B}(a, r_a), |x - y| \leq s \} \quad \text{for } s \in \mathbf{R}, s \geq 0.$$

Choose  $i_a \in \mathcal{P}$  so big that

$$(18) \quad \begin{aligned} & \Delta \|S_{a, i_{a, \mathfrak{h}}} - T_{\mathfrak{h}}\| < 1, \\ & 2L_a \omega_a(L_a r_a \Delta \|S_{a, i_{a, \mathfrak{h}}} - T_{\mathfrak{h}}\|) + L_a^2 \Delta \|S_{a, i_{a, \mathfrak{h}}} - T_{\mathfrak{h}}\| < \min\{1, \varepsilon\}. \end{aligned}$$

Employ 4.2 with  $S_{a, i_a}$ ,  $\varphi_a$ ,  $r_a$ ,  $\tilde{r}_a$  in place of  $S$ ,  $\varphi$ ,  $r$ ,  $\tilde{r}$  to obtain a diffeomorphism  $\rho = \rho_a \in \mathcal{C}^k(\mathbf{R}^n, \mathbf{R}^n)$  and a projection  $p = p_a \in \mathbf{O}^*(n, m)$  satisfying (4), (5), (6).

To finish the construction, we set

$$\rho_\varepsilon(x) = \begin{cases} \rho_a(x) & \text{if } x \in \mathbf{B}(a, r_a) \in \mathcal{B}, \\ x & \text{if } x \in \mathbf{R}^n \sim \bigcup \mathcal{B}. \end{cases}$$

Since  $\mathcal{B}$  is disjointed and each  $\rho_a$  is the identity outside the corresponding ball  $\mathbf{B}(a, r_a) \in \mathcal{B}$ , we see that  $\rho_\varepsilon$  is a well defined diffeomorphism of class  $\mathcal{C}^k$ . Moreover, using (12) and (15), then (5) and (4) together with (16) and finally (6) combined with (17) we obtain

$$\begin{aligned} \mathcal{H}^m(f \circ \rho_\varepsilon[K]) & \leq \mathcal{H}^m(f[K \sim A]) + \mathcal{H}^m(f[(K \cap A) \sim \bigcup \mathcal{B}]) + \sum_{B \in \mathcal{B}} \mathcal{H}^m(f \circ \rho_\varepsilon[K \cap B]) \\ & = \sum_{a \in I} \mathcal{H}^m(f \circ \rho_a[K \cap \mathbf{B}(a, r_a)]) = \sum_{a \in I} \mathcal{H}^m(f \circ \rho_a[K \cap \mathbf{B}(a, r_a) \sim \mathbf{B}(a, \tilde{r}_a)]) \\ & \leq \sum_{a \in I} (2L_a)^m \mathcal{H}^m(K \cap \mathbf{B}(a, r_a) \sim \mathbf{B}(a, \tilde{r}_a)) \leq \varepsilon \sum_{a \in I} \mathcal{H}^m(K \cap \mathbf{B}(a, r_a)) \leq \varepsilon \mathcal{H}^m(K). \end{aligned}$$

Recalling (6) and (18) we see also

$$\sup\{\|D\rho_\varepsilon(x) - \text{id}_{\mathbf{R}^n}\|\} = \sup\{\|D\rho_a(x) - \text{id}_{\mathbf{R}^n}\| : a \in I, x \in \mathbf{B}(a, r_a)\} \leq \varepsilon$$

and

$$\begin{aligned} \sup\{\|\rho_\varepsilon(x) - x\|\} &= \sup\{\|\rho_a(x) - x\| : a \in I, x \in \mathbf{B}(a, r_a)\} \\ &\leq \sup\{\text{Lip}(\rho_a - \text{id}_{\mathbf{R}^n})r_a : a \in I\} < \varepsilon \sup\{r_a : a \in I\} \leq \varepsilon. \quad \square \end{aligned}$$

**4.4 Remark.** If  $k \geq n - m + 1$ , then the Morse–Sard theorem [Fed69, 3.4.3] implies that  $\mathcal{H}^m(f[\{x \in K : \dim \text{im } Df(x) < m\}]) = 0$  and assumption (12) becomes redundant.

**4.5 Corollary.** Set  $g = f \circ \rho_\varepsilon$  and

$$(19) \quad \omega(r) = \sup\{\|Df(x) - Df(y)\| : x, y \in U, |x - y| < r\} \quad \text{for } r > 0.$$

Then for  $x \in \mathbf{R}^n$  we obtain

$$\begin{aligned} \|Dg(x) - Df(x)\| &= \|(Df(\rho_\varepsilon(x)) - Df(x)) \circ D\rho_\varepsilon(x) + Df(x) \circ (D\rho_\varepsilon(x) - \text{id}_{\mathbf{R}^n})\| \\ &\leq 2\omega(\varepsilon) + \|Df(x)\|\varepsilon. \end{aligned}$$

In particular, for  $x \in \mathbf{R}^n$

$$\begin{aligned} \|Dg(x)\|^m &\leq (\|Df(x)\| + \|Dg(x) - Df(x)\|)^m \leq ((1 + \varepsilon)\|Df(x)\| + 2\omega(\varepsilon))^m \\ &\leq 2^{2m-1}\|Df(x)\|^m + 2^{2m-1}\omega(\varepsilon)^m. \end{aligned}$$

## 5 Smooth almost retraction of $\mathbf{R}^n$ onto a cube

In this section we construct, in 5.3, a  $\mathcal{C}^\infty$  function which maps all of  $\mathbf{R}^n$  onto the cube  $Q = [-1, 1]^n$ . This mapping is *not* a retraction because it moves points inside the cube  $Q$ . Its main features are that it is smooth and it preserves all the lower dimensional skeletons of  $Q$  and even the skeletons of the neighbouring dyadic cubes of side length 1. As a corollary of 5.3 we produce, in 5.4, a function which maps a small neighbourhood of  $Q$  onto  $Q$  and is the identity outside a bit larger neighbourhood of  $Q$ . We also carefully track the Lipschitz constants of the mappings.

First we need to introduce some notation to be able to conveniently handle various faces of the cube  $[-1, 1]^n$  and its dyadic neighbours.

**5.1.** Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbf{R}^n$ . Set  $Q = \{x \in \mathbf{R}^n : x \bullet e_j \leq 1 \text{ for } j = 1, 2, \dots, n\} = [-1, 1]^n$ . For  $\kappa = (\kappa_1, \dots, \kappa_n) \in \{-1, 0, 1\}^n$  define

$$\begin{aligned} C_\kappa &= \left\{ x \in \mathbf{R}^n : \begin{array}{l} \text{for } j = 1, \dots, n \\ \text{either } \kappa_j \neq 0 \text{ and } (x \bullet e_j)\kappa_j \geq 1 \\ \text{or } \kappa_j = 0 \text{ and } |x \bullet e_j| < 1 \end{array} \right\}, \\ F_\kappa &= C_\kappa \cap Q, \quad T_\kappa = \text{span}\{e_j : \kappa_j = 0\}, \quad c_\kappa = \sum_{j=1}^n \kappa_j e_j. \end{aligned}$$

Observe that the sets  $C_\kappa$  for  $\kappa \in \{-1, 0, 1\}$  are convex, have nonempty interiors, are pairwise disjoint, and form a partition of  $\mathbf{R}^n$ , i.e.,

$$\bigcup \{C_\kappa : \kappa \in \{-1, 0, 1\}\} = \mathbf{R}^n \quad \text{and} \quad C_\kappa \cap C_\lambda = \emptyset \text{ for } \lambda \neq \kappa.$$

For  $\kappa \in \{-1, 0, 1\}$  we have  $\dim(T_\kappa) = \mathcal{H}^0(\{j : \kappa_j = 0\})$ , and  $F_\kappa$  is a  $\dim(T_\kappa)$  dimensional face of  $Q$  lying in the affine space  $c_\kappa + T_\kappa$ , and  $F_\kappa$  is relatively open in  $c_\kappa + T_\kappa$ , and  $c_\kappa$  is the centre of  $F_\kappa$ . In particular,  $C_{(0,0,\dots,0)} = F_{(0,0,\dots,0)} = \text{Int}(Q)$ .

For  $\lambda \in \{-2, -1, 1, 2\}^n$  define

$$R_\lambda = \left\{ x \in \mathbf{R}^n : \begin{array}{l} \text{for } j = 1, \dots, n \\ \text{either } |\lambda_j| = 1 \text{ and } 1 \leq (x \bullet e_j) \lambda_j \leq 2 \\ \text{or } |\lambda_j| = 2 \text{ and } 0 \leq (x \bullet e_j) \lambda_j \leq 2 \end{array} \right\}.$$

Note that  $R_\lambda$  is isometric to  $[0, 1]^n$  and if  $\lambda \notin \{-2, 2\}^n$ , then  $R_\lambda$  is one of the neighbouring cubes of  $Q$  with side length equal to half the side length of  $Q$ .

Set

$$f_\kappa(x) = (T_\kappa)_\sharp x + c_\kappa \quad \text{and} \quad f(x) = \sum_{\kappa \in \{-1, 0, 1\}^n} \mathbb{1}_{C_\kappa}(x) f_\kappa(x) \quad \text{for } x \in \mathbf{R}^n,$$

where  $\mathbb{1}_{C_\kappa}$  is the characteristic function of  $C_\kappa$ .

**5.2 Remark.** Observe that  $f$  is simply the nearest point projection from  $\mathbf{R}^n$  onto  $Q$ . Since  $Q$  is convex it has infinite reach and [Fed59, 4.8(4)(8)] shows that  $f$  is Lipschitz continuous. However, we shall need the above decomposition of  $f$  to be able to effectively smoothen the singularities, see 5.3.

In the next lemma we construct a smooth mapping from  $\mathbf{R}^n$  onto  $[-1, 1]^n$ . This is achieved by post-composing the nearest point projection  $f$  with a smooth function which has zero derivative exactly in the directions in which the derivative of  $f$  is undefined.

**5.3 Lemma.** Let  $e_1, \dots, e_n, f, C_\kappa, F_\kappa, Q$  be as in 5.1. Assume  $s : \mathbf{R} \rightarrow \mathbf{R}$  is a function of class  $\mathcal{C}^\infty$  such that  $D^i s(-1) = 0$  and  $D^i s(1) = 0$  for  $i \in \mathcal{P}$ . Define  $h(x) = \sum_{j=1}^n s(x \bullet e_j) e_j$  for  $x \in \mathbf{R}^n$ . Then

- (a)  $g = h \circ f$  is of class  $\mathcal{C}^\infty$  with  $D^i g = D^i h \circ f$  for  $i \in \mathcal{P}$ .
- (b) If  $s$  is monotone increasing and  $s(t) = t$  for  $t \in \{-2, -1, 0, 1, 2\}$ , then for each  $\kappa \in \{-1, 0, 1\}^n$  and  $\lambda \in \{-2, -1, 1, 2\}^n$  if  $C_\kappa, F_\kappa, T_\kappa, c_\kappa, R_\lambda$  are as in 5.1, then

$$\begin{aligned} g[C_\kappa] &= F_\kappa, \quad g|_{F_\kappa} : F_\kappa \rightarrow F_\kappa \quad \text{is a homeomorphism,} \\ g[T_\kappa] &\subseteq T_\kappa, \quad g[R_\lambda] \subseteq R_\lambda, \quad g[c_\kappa + T_\kappa] \subseteq c_\kappa + T_\kappa. \end{aligned}$$

- (c) Let  $\varepsilon \in (0, 1)$ . Assume  $s$  satisfies  $0 \leq s'(t) \leq 1 + \varepsilon$  and  $|s(t) - t| \leq \varepsilon$  for  $t \in \mathbf{R}$  and  $s'(t) > 0$  for  $t \in \mathbf{R} \setminus \{-1, 1\}$ . Then

$$\begin{aligned} |g(x) - x| &\leq (1 + \sqrt{n})\varepsilon \quad \text{for } x \in Q + \mathbf{B}(0, \varepsilon), \\ \text{Lip}(g) &= \text{Lip}(h|_Q) \leq 1 + \varepsilon, \quad \text{Lip}(g - \text{id}_{\mathbf{R}^n}) \leq 1, \\ \text{Lip}(g|_K - \text{id}_K) &< 1 \quad \text{for each compact set } K \subseteq \text{Int } Q. \end{aligned}$$

*Proof.* Since  $s'(1) = 0 = s'(-1)$  we have

$$(20) \quad Dh(y)u = \sum_{j:\lambda_j=0} s'(y \bullet e_j)(u \bullet e_j)e_j = Dh(y)((T_\lambda)_\natural u) \in T_\lambda \quad \text{for } y \in C_\lambda.$$

First, assume  $x \in C_\lambda$  and  $\dim(T_\lambda) > 0$ . Define  $p = (T_\lambda)_\natural$  and  $q = (T_\lambda^\perp)_\natural$ . Observe that for  $j = 1, 2, \dots, n$

$$(21) \quad \begin{aligned} \lambda_j &= \text{sgn}(x \bullet e_j) \quad \text{if and only if} \quad |x \bullet e_j| \geq 1, \\ \text{and} \quad \lambda_j &= 0 \quad \text{if and only if} \quad |x \bullet e_j| < 1. \end{aligned}$$

Since  $\dim(T_\lambda) > 0$  there exists  $j \in \{1, 2, \dots, n\}$  such that  $\lambda_j = 0$ . Let  $u \in \mathbb{R}^n$  be such that

$$(22) \quad 0 < |u| < \min\{1 - |x \bullet e_j| : \lambda_j = 0\} < 1.$$

Let  $\kappa \in \{-1, 0, 1\}^n$  be defined by

$$(23) \quad \begin{aligned} \kappa_j &= \text{sgn}((x + qu) \bullet e_j) \quad \text{if and only if} \quad |(x + qu) \bullet e_j| \geq 1, \\ \text{and} \quad \kappa_j &= 0 \quad \text{if and only if} \quad |(x + qu) \bullet e_j| < 1. \end{aligned}$$

Then  $x + tqu \in C_\kappa$  for all  $t \in (0, 1)$  because  $C_\kappa$  is convex; hence,  $x \in \text{Clos}(C_\kappa)$ . Moreover, using (21), (22), (23), we see that for  $j = 1, 2, \dots, n$

$$\begin{aligned} \lambda_j = 0 &\quad \text{implies} \quad |(x + qu) \bullet e_j| = |x \bullet e_j| < 1 \quad \text{implies} \quad \kappa_j = 0, \\ \lambda_j \neq 0 &\quad \text{implies} \quad ((x + qu) \bullet e_j)\lambda_j = (x \bullet e_j)\lambda_j + (u \bullet e_j)\lambda_j > 0; \\ &\quad \text{hence, either} \quad \kappa_j = \lambda_j \neq 0 \quad \text{or} \quad \kappa_j = 0 \text{ and } \lambda_j \neq 0. \end{aligned}$$

Therefore,  $T_\lambda \subseteq T_\kappa$ . Recalling  $x \in \text{Clos}(C_\kappa) \cap C_\lambda$ , we see that if  $j \in \{1, 2, \dots, n\}$  and  $\lambda_j \neq \kappa_j$ , then  $(x \bullet e_j) = \lambda_j$ . Thus,

$$(24) \quad (T_\kappa)_\natural x - (T_\lambda)_\natural x = \sum_{j:\kappa_j=0} (x \bullet e_j)e_j - \sum_{j:\lambda_j=0} (x \bullet e_j)e_j = \sum_{j:\kappa_j=0, \lambda_j \neq 0} \lambda_j e_j = c_\lambda - c_\kappa.$$

Using (24) we derive

$$f(x + qu) - f(x) = f_\kappa(x + qu) - f_\lambda(x) = (T_\kappa)_\natural(qu) \in T_\kappa \cap T_\lambda^\perp.$$

Moreover, since  $pu \in T_\kappa$  and  $x + qu \in C_\kappa$  and, by (22),  $x + qu + pu \in C_\kappa$  we obtain

$$f(x + qu + pu) - f(x + qu) = (T_\kappa)_\natural(x + qu + pu) - (T_\kappa)_\natural(x + qu) = pu \in T_\lambda.$$

Thus,

$$\xi = f(x + u) - f(x) = f(x + qu + pu) - f(x + qu) + f(x + qu) - f(x) = pu + (T_\kappa)_\natural(qu)$$

and, recalling (20),

$$\begin{aligned} |g(x + u) - g(x) - Dh(f(x))u| &= |h(f(x) + \xi) - h(f(x)) - Dh(f(x))(pu)| \\ &= |h(f(x) + \xi) - h(f(x)) - Dh(f(x))(\xi)| \end{aligned}$$

Since  $|\xi| \leq \text{Lip}(f)|u|$  and  $h$  is of class  $\mathcal{C}^1$  we obtain

$$\lim_{u \rightarrow 0} |u|^{-1} |g(x+u) - g(x) - Dh(f(x))u| = 0.$$

This shows that  $Dg(x) = Dh(f(x))$  in case  $\dim(T_\lambda) > 0$ .

Now we shall deal with the case when  $x \in C_\lambda$  and  $\dim(T_\lambda) = 0$ . This means that  $f(x) \in F_\lambda$  is one of the vertexes of  $Q$ . Since  $h$  is of class  $\mathcal{C}^1$  and  $\text{Lip}(f) < \infty$  (see 5.2) and  $Dh(f(x)) = 0$  by (20), we get

$$|g(x+u) - g(x)| = |h(f(x+u)) - h(f(x)) - Dh(f(x))(f(x+u) - f(x))|.$$

Hence, in this case we also get  $Dg(x) = Dh(f(x))$

Now we know that  $Dg(x) = Dh(f(x))$  for all  $x \in \mathbf{R}^n$  and since  $f$  is continuous we see that  $g$  is of class  $\mathcal{C}^1$ . Repeating the whole argument with  $g = h \circ f$  replaced by  $Dg = Dh \circ f$  and proceeding by induction we see that  $g$  is of class  $\mathcal{C}^\infty$  so (a) is proven.

Item (b) readily follows from the definition of  $g$ .

Consider now  $\varepsilon$  and  $s$  as in (c). For  $x \in Q + \mathbf{B}(0, \varepsilon)$  we have

$$\begin{aligned} |g(x) - x| &\leq |f(x) - x| + |h(f(x)) - f(x)| \\ &\leq \varepsilon + \left( \sum_{i=1}^n (s(f(x) \bullet e_i) - f(x) \bullet e_i)^2 \right)^{1/2} \leq (1 + \sqrt{n})\varepsilon. \end{aligned}$$

For  $y \in \mathbf{R}^n$  and  $u \in \mathbf{R}^n$  with  $|u| = 1$

$$|Dh(y)u|^2 = \sum_{i=1}^n s'(y \bullet e_i)^2 (u \bullet e_i)^2 \leq 1 + \varepsilon; \quad \text{hence,} \quad \text{Lip}(g) = \text{Lip}(h|_Q) \leq 1 + \varepsilon.$$

For any  $y \in Q$  and  $u \in \mathbf{R}^n$ , recalling  $-1 \leq s'(t) - 1 \leq \varepsilon < 1$  for  $t \in \mathbf{R}$ , we have

$$|Dh(y)u - u|^2 = \sum_{i=1}^n (s'(y \bullet e_i) - 1)^2 (u \bullet e_i)^2 \leq 1; \quad \text{hence,} \quad \text{Lip}(g - \text{id}_{\mathbf{R}^n}) \leq 1.$$

For  $K \subseteq \text{Int } Q$  compact and  $y \in K$  we have  $-1 < s'(y \bullet e_i) - 1 < \varepsilon < 1$  so  $|Dh(y)u - u|^2 < 1$  whenever  $u \in \mathbf{R}^n$  satisfies  $|u| = 1$ . Consequently,  $\text{Lip}(g|_K - \text{id}_K) < 1$ .  $\square$

Next, using 5.3, we construct another function which maps some neighbourhood of  $Q = [-1, 1]^n$  onto  $Q$  and is the identity a bit further away from  $Q$ . To this end we put  $Q$  inside a convex open set  $V$  with smooth boundary and use the distance from  $V$ , which is smooth away from the boundary of  $V$ , to interpolate between the mapping constructed in 5.3 and the identity.

**5.4 Corollary.** *Let  $n \in \mathcal{P}$ . For each  $\varepsilon \in (0, 1)$  there exists a map  $l : \mathbf{R}^n \rightarrow \mathbf{R}^n$  of class  $\mathcal{C}^\infty$  such that if  $\Gamma = 16\sqrt{n}$ , then*

(a)  $l(x) = x$  for  $x \in \mathbf{R}^n$  satisfying  $\text{dist}(x, Q) > \varepsilon$ .

(b) For each  $\kappa \in \{-1, 0, 1\}^n$  and  $\lambda \in \{-2, -1, 1, 2\}^n$  if  $C_\kappa, F_\kappa, T_\kappa, c_\kappa, R_\lambda$  are as in 5.1, then

$$\begin{aligned} l[T_\kappa] &\subseteq T_\kappa, \quad l[F_\kappa] \subseteq F_\kappa, \quad l[C_\kappa] \subseteq C_\kappa, \quad l[c_\kappa + T_\kappa] \subseteq c_\kappa + T_\kappa, \\ l[R_\lambda] &\subseteq R_\lambda, \quad l[\{x \in C_\kappa : \text{dist}(x, Q) \leq \varepsilon/\Gamma\}] \subseteq F_\kappa. \end{aligned}$$



- (c)  $l|_{\text{Int } Q} : \text{Int } Q \rightarrow \text{Int } Q$  is a diffeomorphism such that for each compact set  $K \subseteq \text{Int } Q$  we have  $\text{Lip}(l|_K - \text{id}_K) < 1$ .
- (d)  $\text{Lip}(l|_Q - \text{id}_Q) \leq 1$  and  $\text{Lip}(l|_Q) \leq 1 + \varepsilon$ .
- (e)  $\text{Lip}(l) < \Gamma$ .
- (f)  $|l(x) - x| \leq \varepsilon$  for  $x \in \mathbf{R}^n$ .
- (g)  $\text{dist}(l(x), Q) \leq \text{dist}(x, Q)$  for  $x \in \mathbf{R}^n$ .

*Proof.* Let  $n \in \mathcal{P}$  and  $\varepsilon \in (0, 1)$ . Set  $\iota = \varepsilon/(2(1 + \sqrt{n}))$ . Let  $\alpha : \mathbf{R} \rightarrow \mathbf{R}$  be map of class  $\mathcal{C}^\infty$  such that

$$\alpha(t) = 0 \quad \text{for } t \leq 0, \quad \alpha(t) = 1 \quad \text{for } t \geq 1, \quad 0 < \alpha'(t) \leq 1 + \iota \quad \text{for } t \in (0, 1).$$

Let  $s : \mathbf{R} \rightarrow \mathbf{R}$  be a homeomorphism of class  $\mathcal{C}^\infty$  such that

$$\begin{aligned} 0 \leq s'(t) \leq 1 + \iota \quad \text{and} \quad |s(t) - t| \leq \iota \quad \text{for } t \in \mathbf{R}, \quad s(t) = t \quad \text{for } t \in \{-2, -1, 0, 1, 2\}, \\ s'(t) > 0 \quad \text{for } t \in \mathbf{R} \setminus \{-1, 1\}, \quad D^j s(-1) = 0 = D^j s(1) \quad \text{for each } j \in \mathcal{P}. \end{aligned}$$

Choose an open convex set  $V \subseteq \mathbf{R}^n$  such that  $Q + \mathbf{B}(0, \iota/4) \subseteq V \subseteq Q + \mathbf{B}(0, \iota/2)$  and  $\text{Bdry } V$  is a submanifold of  $\mathbf{R}^n$  of class  $\mathcal{C}^\infty$ . Define  $g$  as in 5.3 using  $s$  and set

$$\delta(x) = \text{dist}(x, V), \quad l(x) = g(x) + (x - g(x))\alpha(2\delta(x)/\iota).$$

Since  $V$  is convex and  $\text{Bdry } V$  is of class  $\mathcal{C}^\infty$ , we see that  $l$  is of class  $\mathcal{C}^\infty$ . Clearly  $l(x) = x$  whenever  $\text{dist}(x, Q) \geq \varepsilon \geq \iota$  which establishes (a).

*Proof of (b).* Since  $T_\kappa, F_\kappa, C_\kappa, c_\kappa + T_\kappa, R_\lambda$  are convex the inclusions  $l[T_\kappa] \subseteq T_\kappa, l[F_\kappa] \subseteq F_\kappa, l[C_\kappa] \subseteq C_\kappa, l[c_\kappa + T_\kappa] \subseteq c_\kappa + T_\kappa, l[R_\lambda] \subseteq R_\lambda$  readily follow from 5.3(b). For  $x \in V$  we have  $l(x) = g(x)$  so, noting  $\iota/4 \geq \varepsilon/(16\sqrt{n})$ , we see that  $l(x) = g(x) \in F_\kappa$  whenever  $x \in C_\kappa$  and  $\text{dist}(x, Q) \leq \varepsilon/\Gamma$ .

Employing [Fed59, 4.8(3)], we see that  $\text{Lip}(\delta) = 1$ ; hence, we obtain for  $x \in Q + \mathbf{B}(0, \varepsilon)$

$$\|Dl(x)\| = \|Dg(x)\| + \|Dg(x) - \text{id}_{\mathbf{R}^n}\| + |x - g(x)|(1 + \iota)2/\iota$$

Items (c) and (d) follow immediately from 5.3(c) noting that  $l(x) = g(x)$  for  $x \in Q$ . Recalling 5.3(c) we have

$$\text{Lip}(l) \leq 1 + \iota + 1 + 2(1 + \sqrt{n})\iota(1 + \iota)/\iota \leq 11\sqrt{n}$$

so (e) holds. For  $x \in \mathbf{R}^n$  with  $\text{dist}(x, Q) \leq \varepsilon$  we have

$$|l(x) - x| \leq |g(x) - x| + |x - g(x)| \leq 2(1 + \sqrt{n})\iota = \varepsilon,$$

which proves (f). To verify (g) note that  $l(x) \in \text{conv}\{x, g(x)\}$  and  $g(x) \in Q$  for  $x \in \mathbf{R}^n$ .  $\square$

## 6 Central projections

Here we study analytic properties of the central projection from the origin onto the boundary of a bounded convex set  $V$  containing 0. In 6.4 we derive formulas and estimates for the derivative of such projection in terms of the position of the origin with respect to the

boundary  $\text{Bdry } V$  and the shape of  $\text{Bdry } V$ . Then in 6.5 we interpolate between a central projection and the identity to get a map which acts as the central projection inside  $V$  and is the identity outside given neighbourhood of  $V$ .

We start by deriving a formula for the derivative of the central projection onto the boundary of a half-space.

**6.1 Lemma.** *Let  $\nu, y \in \mathbf{R}^n$  be such that  $|\nu| = 1$  and  $\nu \bullet y > 0$ . Define*

$$\begin{aligned} U &= \{z \in \mathbf{R}^n : \nu \bullet z > 0\}, \\ s(z) &= \sup\{t \in \mathbf{R} : tz \in U\} \quad \text{for } z \in U, \\ \pi(z) &= s(z)z \quad \text{for } z \in U. \end{aligned}$$

*Then  $s : U \rightarrow \mathbf{R}$  and  $\pi : U \rightarrow \mathbf{R}^n$  are maps of class  $\mathcal{C}^\infty$  and for  $z \in U$ ,  $u \in \mathbf{R}^n$*

$$s(z) = \frac{\nu \bullet y}{\nu \bullet z}, \quad Ds(z)u = -\frac{(\nu \bullet y)(\nu \bullet u)}{(\nu \bullet z)^2}, \quad \|D\pi(z)\| \leq \frac{1}{|z|} \left( \frac{\nu \bullet y}{\nu \bullet \frac{z}{|z|}} + \frac{\nu \bullet y}{(\nu \bullet \frac{z}{|z|})^2} \right).$$

*Proof.* Let  $T = \text{span}\{\nu\}^\perp$ . Observe that  $U$  is the half-space in  $\mathbf{R}^n$  cut by  $T$ , and  $\pi(z) \in y + T$  for  $z \in U$ , and  $\pi$  is the central projection onto  $y + T$  with centre at the origin. Since  $\pi(z) = s(z)z \in y + T$  we have

$$T^\perp_{\mathfrak{h}}(s(z)z - y) = 0; \quad \text{hence,} \quad s(z) = \frac{|T^\perp_{\mathfrak{h}} y|}{|T^\perp_{\mathfrak{h}} z|} = \frac{\nu \bullet y}{\nu \bullet z}.$$

A straightforward computation proves the other assertions.  $\square$

**6.2 Definition.** Let  $V \subseteq \mathbf{R}^n$  be an open bounded convex set such that  $0 \in V$ . We say that a pair of maps  $p : \mathbf{R}^n \sim \{0\} \rightarrow \mathbf{R}^n$  and  $t : \mathbf{R}^n \sim \{0\} \rightarrow (0, \infty)$  define the *central projection onto  $\text{Bdry } V$*  if

$$p(x) = t(x)x \quad \text{and} \quad t(x) = \sup\{t > 0 : tx \in V\} \quad \text{for } x \in \mathbf{R}^n \sim \{0\}.$$

In the next lemma we prove that the derivative at some point  $x$  of the central projection onto the boundary of a convex set  $V$  depends only on the affine tangent plane of  $\text{Bdry } V$  at  $x$  (assuming it exists) and, actually, coincides with the derivative of the central projection onto that tangent plane.

**6.3 Lemma.** *Let  $V \subseteq \mathbf{R}^n$  be an open bounded convex set with  $0 \in V$ . Assume  $y \in \text{Bdry } V$ , and  $\text{Tan}(\text{Bdry } V, y) \in \mathbf{G}(n, n-1)$ , and  $\nu \in \text{Tan}(\text{Bdry } V, y)^\perp$  is the outward pointing unit normal to  $\text{Bdry } V$  at  $y$ ; in particular  $|\nu| = 1$  and  $\nu \bullet y > 0$ . Suppose  $U, s, \pi$  are defined as in 6.1 and  $p, t$  define the central projection onto  $\text{Bdry } V$ .*

*If  $x \in \mathbf{R}^n \sim \{0\}$  satisfies  $p(x) = y$ , then  $p$  is differentiable at  $x$  and*

$$x \in U, \quad \pi(x) = p(x) = y, \quad D\pi(x) = Dp(x).$$

*Proof.* Fix  $x \in \mathbf{R}^n \sim \{0\}$  such that  $p(x) = y$ . Set

$$\eta = \frac{x}{|x|}, \quad S = \text{span}\{\eta\}^\perp, \quad T = \text{span}\{\nu\}^\perp, \quad \theta = \|S_{\mathfrak{h}} - T_{\mathfrak{h}}\| = (1 - (\eta \bullet \nu)^2)^{1/2} < 1.$$

Let  $\delta \in \mathbf{R}$  satisfy  $0 < \delta \leq 2^{-10}(\eta \bullet \nu)^{1/2}$ . For  $h \in \mathbf{R}^n$  with  $|h| \leq \delta|x|$  define

$$\gamma_h = \frac{x+h}{|x+h|}, \quad Z_h = \text{span}\{\gamma_h\},$$

and note that

$$\gamma_h \bullet \eta = \left(1 + \frac{|S_{\natural}h|^2}{|S_{\natural}^{\perp}(x+h)|^2}\right)^{-1/2} \geq \left(1 + \frac{\delta^2}{(1-\delta)^2}\right)^{-1/2} > 0,$$

$$\gamma_h \bullet \nu \geq \eta \bullet \nu - |\gamma_h - \eta| = \eta \bullet \nu - 2(1 - \gamma_h \bullet \eta) \geq \eta \bullet \nu - \frac{\delta^2}{(1-\delta)^2} \geq (1 - 2^{-18})\eta \bullet \nu > 0.$$

For  $h \in \mathbf{R}^n$  with  $|h| \leq \delta|x|$  we have

$$\|S_{\natural} - (Z_h^{\perp})_{\natural}\| = |S_{\natural}\gamma_h| = (1 - (\eta \bullet \gamma_h)^2)^{1/2} < 1,$$

$$\|T_{\natural} - (Z_h^{\perp})_{\natural}\| = |T_{\natural}\gamma_h| = (1 - (\nu \bullet \gamma_h)^2)^{1/2} < 1;$$

hence, we can define

$$\lambda = \sup\{\|S_{\natural} - (Z_h^{\perp})_{\natural}\| : h \in \mathbf{R}^n, |h| \leq \delta|x|\} < 1,$$

$$\varphi = \sup\{\|T_{\natural} - (Z_h^{\perp})_{\natural}\| : h \in \mathbf{R}^n, |h| \leq \delta|x|\} < 1.$$

Next, set

$$\beta(r) = \frac{1}{r} \sup\{|T_{\natural}^{\perp}(z-y)| : z \in \text{Bdry } V \cap \mathbf{B}(y, r)\}.$$

Since  $T = \text{Tan}(\text{Bdry } V, y) \in \mathbf{G}(n, n-1)$  we know that  $\beta(r) \rightarrow 0$  as  $r \downarrow 0$  and there exists  $0 < r_0 < 1$  such that  $\beta(r) \leq \frac{1}{2}(1-\theta)$  for  $0 < r < r_0$ .

Observe that  $p$  is continuous at  $x$ . If it were not, there would exist a sequence  $h_i \in \mathbf{R}^n$  such that  $|h_i| \rightarrow 0$  as  $i \rightarrow \infty$  but  $y_i = p(x + h_i)$  would not converge to  $y = p(x)$ . Then  $y_i$  would be in the cone  $\{tw : t > 0, |x-w| \leq |h_i|\}$  so, since  $V$  is bounded, one could choose a subsequence of  $y_i$  which would converge to some point  $y_0 \in S^{\perp}$  and  $y_0 \neq y$ . Since  $V$  is convex, this would imply that  $\{ty + (1-t)y_0 : 0 \leq t \leq 1\} \subseteq \text{Bdry } V$ . This, in turn, would mean that  $\eta \in T = \text{Tan}(\text{Bdry } V, y)$  which is impossible because  $|T_{\natural}^{\perp}\eta| = \eta \bullet \nu > 0$ .

Knowing that  $p$  is continuous we can find  $\rho_0 > 0$  such that  $|p(x+h) - p(x)| \leq r_0$  whenever  $|h| \leq \rho_0$ . Fix  $h \in \mathbf{R}^n$  with  $|h| \leq \min\{\delta|x|, \rho_0\}$  and let  $b = p(x+h)$ . Set

$$\Gamma = 2\left(1 + \frac{\theta}{1-\varphi}\right)\left(\frac{|y|}{|x|} + \frac{1}{2|x|}\right).$$

We shall show that  $|b-y| = |p(x+h) - p(x)| \leq \Gamma|h|$ .

Let  $a, z \in \mathbf{R}^n$  be such that

$$\{z\} = (b+T) \cap S^{\perp}, \quad \{a\} = (z+S) \cap \{tb : t > 0\}; \quad \text{then} \quad b \in z+T.$$

Since  $(Z_h^{\perp})_{\natural}(b-a) = 0$  and  $S_{\natural}^{\perp}(z-a) = 0$  and  $T_{\natural}^{\perp}(b-z) = 0$  we obtain

$$\begin{aligned} |b-a| &\leq |T_{\natural}(b-a)| + |T_{\natural}^{\perp}(b-a)| \\ &\leq |(T_{\natural} - (Z_h^{\perp})_{\natural})(b-a)| + |T_{\natural}^{\perp}(b-z)| + |(T_{\natural}^{\perp} - S_{\natural}^{\perp})(z-a)| \leq \varphi|b-a| + \theta|a-z|. \end{aligned}$$

Thus

$$|b - a| \leq \frac{\theta}{1 - \varphi} |a - z| \quad \text{and} \quad |b - z| \leq |b - a| + |a - z| \leq \left(1 + \frac{\theta}{1 - \varphi}\right) |a - z|.$$

Directly from the definition of  $a$  it follows that

$$|a - z| = \frac{|z|}{|x|} |S_{\natural} h| \leq \frac{|z|}{|x|} |h|; \quad \text{hence,} \quad |b - z| \leq \left(1 + \frac{\theta}{1 - \varphi}\right) \frac{|z|}{|x|} |h|.$$

Recall that  $|h| \leq \rho_0$  so  $|b - y| \leq r_0 < 1$  so  $\beta(|b - y|) \leq \frac{1}{2}(1 - \theta)$  and we can write

$$|z - y| \leq |T^{\perp}_{\natural}(z - y)| + |T_{\natural} S^{\perp}_{\natural}(z - y)| \leq \beta(|b - y|) |b - y| + \theta |z - y|; \quad \text{so} \quad |z| \leq |y| + \frac{1}{2}.$$

In consequence

$$|b - y| \leq \left(1 + \frac{\theta}{1 - \varphi}\right) \left(\frac{|y|}{|x|} + \frac{1}{2|x|}\right) |h| + \frac{1}{2} |b - y| \quad \text{so} \quad |b - y| \leq \Gamma |h|.$$

Let  $h \in \mathbf{R}^n$  be such that  $|h| \leq \min\{\delta|x|, \rho_0\}$ . Now we are ready to estimate  $|p(x + h) - \pi(x + h)|$ . Set  $u = p(x + h) - \pi(x + h)$  and observe that

$$\begin{aligned} |T^{\perp}_{\natural}(p(x + h) - y)| &\leq \beta(\Gamma|h|) \Gamma|h|, \quad u \in Z_h, \quad |T_{\natural} u| = |T_{\natural} Z_{\natural} u| \leq \varphi|u|, \\ |T^{\perp}_{\natural} u| &\leq |T^{\perp}_{\natural}(p(x + h) - y)| + |T^{\perp}_{\natural}(y - \pi(x + h))| = |T^{\perp}_{\natural}(p(x + h) - y)| \leq \beta(\Gamma|h|) \Gamma|h|, \\ |u| &\leq |T_{\natural} u| + |T^{\perp}_{\natural} u| \leq \varphi|u| + \beta(\Gamma|h|) \Gamma|h|, \\ |p(x + h) - \pi(x + h)| &= |u| \leq \frac{1}{1 - \varphi} \beta(\Gamma|h|) \Gamma|h|. \end{aligned}$$

It is clear from the definitions that  $p(x) = \pi(x) = y$ . In consequence

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{|p(x + h) - p(x) - D\pi(x)h|}{|h|} \\ \leq \lim_{h \rightarrow 0} \frac{|\pi(x + h) - \pi(x) - D\pi(x)h|}{|h|} + \frac{|p(x + h) - \pi(x + h)|}{|h|} = 0, \end{aligned}$$

which shows that  $p$  is differentiable at  $x$  and  $Dp(x) = D\pi(x)$ .  $\square$

**6.4 Corollary.** Suppose  $V \subseteq \mathbf{R}^n$  is an open bounded convex set, and  $0 \in V$ , and  $\text{Bdry } V$  is an  $(n-1)$  dimensional submanifold of  $\mathbf{R}^n$  of class  $\mathcal{C}^{\infty}$ , and  $p, t$  define the central projection onto  $\text{Bdry } V$ , and  $\nu(y)$  is the outward pointing unit normal to  $\text{Bdry } V$  at  $y$  for  $y \in \text{Bdry } V$ . Then  $p$  and  $t$  are of class  $\mathcal{C}^{\infty}$  and

$$Dt(x)u = -\frac{(\nu(p(x)) \bullet p(x))(\nu(p(x)) \bullet u)}{(\nu(p(x)) \bullet x)^2}, \quad \|Dp(x)\| \leq \frac{|p(x)|}{|x|} \left(1 + \frac{1}{\nu(p(x)) \bullet \frac{x}{|x|}}\right).$$

for  $x \in \mathbf{R}^n \sim \{0\}$  and  $u \in \mathbf{R}^n$ .

*Proof.* Since  $\text{Bdry } V$  is of class  $\mathcal{C}^1$  we can apply 6.3 and 6.1 at any single point  $x \in \mathbf{R}^n \sim \{0\}$  to see that

$$Dp(x)u = \frac{\nu(p(x)) \bullet p(x)}{\nu(p(x)) \bullet x} u - \frac{(\nu(p(x)) \bullet p(x))(\nu(p(x)) \bullet u)}{(\nu(p(x)) \bullet x)^2} x$$

for  $u \in \mathbf{R}^n$ . Noting  $t(x) = p(x) \bullet x / (x \bullet x)$  one derives the formula for  $Dt(x)$ . This shows that  $p$  and  $t$  are of class  $\mathcal{C}^1$ . Since  $\nu$  is of class  $\mathcal{C}^{\infty}$ , proceeding by induction, we see that  $p$  and  $t$  are of class  $\mathcal{C}^{\infty}$ .  $\square$

Next, we construct a map which interpolates between the central projection onto  $\text{Bdry } V$  and identity outside some neighbourhood of  $V$ .

**6.5 Corollary.** *Let  $\varepsilon \in (0, 1)$  and  $V \subseteq \mathbf{R}^n$  be open convex with  $0 \in V$  and  $p, t$  define the central projection onto  $\text{Bdry } V$ . Assume  $\text{Bdry } V$  is an  $n - 1$  dimensional submanifold of  $\mathbf{R}^n$  of class  $\mathcal{C}^\infty$ . Then there exists a map  $q : \mathbf{R}^n \sim \{0\} \rightarrow \mathbf{R}^n$  of class  $\mathcal{C}^\infty$  such that*

- (a)  $q(x) = x$  for  $x \in \mathbf{R}^n \sim V$ .
- (b)  $q(x) = p(x)$  for  $x \in V \sim \{0\}$  with  $\text{dist}(x, \mathbf{R}^n \sim V) \geq \varepsilon$ .
- (c) For each  $x \in \mathbf{R}^n \sim \{0\}$  there exists  $t \in [1, \infty)$  such that  $q(x) = tx$ .
- (d)  $q(x) \in \text{conv}\{x, p(x)\}$  for each  $x \in \mathbf{R}^n \sim \{0\}$ .
- (e)  $|q(x) - x| \leq |p(x) - x|$  whenever  $x \in V \sim \{0\}$ .
- (f)  $\|Dq(x)\| \leq 5|p(x)||x|^{-1}\Delta$  for  $x \in V \sim \{0\}$ , where  $\Delta = \inf\{\nu(y) \bullet \frac{y}{|y|} : y \in \text{Bdry } V\}^{-1}$ .

*Proof.* Set

$$\begin{aligned} \iota &= \min\{\varepsilon, \inf\{\text{dist}(x, \mathbf{R}^n \sim V) : x \in V, t(x) \geq 1 + \varepsilon\}\}, \\ \delta &= \inf\{t(x) : x \in V, \text{dist}(x, \mathbf{R}^n \sim V) \geq \iota\}. \end{aligned}$$

Then for  $x \in \mathbf{R}^n \sim \{0\}$

$$(25) \quad 1 < t(x) < \delta \quad \text{implies} \quad \text{dist}(x, \mathbf{R}^n \sim V) < \iota \leq \varepsilon,$$

$$(26) \quad \text{dist}(x, \mathbf{R}^n \sim V) < \iota \quad \text{implies} \quad t(x) - 1 < \varepsilon,$$

$$\text{dist}(x, \mathbf{R}^n \sim V) \geq \iota \quad \text{implies} \quad t(x) \geq \delta.$$

Choose  $\alpha : \mathbf{R} \rightarrow \mathbf{R}$  of class  $\mathcal{C}^\infty$  such that

$$\begin{aligned} \alpha(t) &= t \quad \text{for } t \geq \delta, \quad \alpha(t) \leq t \quad \text{for } t \geq 1, \\ \alpha(t) &= 1 \quad \text{for } t \leq 1, \quad 0 \leq \alpha'(t) \leq 2 \quad \text{for } t \in \mathbf{R}. \end{aligned}$$

Set  $q(x) = \alpha(t(x))x$ .

Clearly  $q : \mathbf{R}^n \sim \{0\} \rightarrow \mathbf{R}^n$  is of class  $\mathcal{C}^\infty$  and  $q(x) \in \text{conv}\{x, p(x)\}$  for  $x \in \mathbf{R}^n \sim \{0\}$  and  $q(x) = x$  for  $x \in \mathbf{R}^n \sim V$ . Moreover,  $q(x) = p(x)$  for  $x \in V \sim \{0\}$  satisfying  $\text{dist}(x, \mathbf{R}^n \sim V) \geq \varepsilon$ , which follows by (26). For  $x \in V \sim \{0\}$  with  $0 < \text{dist}(x, \mathbf{R}^n \sim V) \leq \iota$  we have, using (25),

$$|q(x) - x| = |x|(\alpha(t(x)) - 1) \leq |x|(t(x) - 1) = |p(x) - x|$$

For  $x \in V \sim \{0\}$ , using the formulas from 6.4 and the identity  $t(x) = |p(x)||x|^{-1}$ , we get

$$\begin{aligned} \|Dp(x)\| &\leq |p(x)||x|^{-1}(1 + \Delta) \leq 2|p(x)||x|^{-1}\Delta, \\ |Dq(x)u| &\leq |\alpha'(t(x))Dp(x)u| + |\alpha'(t(x))t(x)u| + |\alpha(t(x))u| \quad \text{for } u \in \mathbf{R}^n, |u| = 1, \\ \|Dq(x)\| &\leq 5|p(x)||x|^{-1}\Delta. \end{aligned}$$

□

## 7 Smooth deformation theorem

Here we prove a version of Federer–Fleming projection theorem suited for our purposes. The proof follows the scheme of [Fed69, 4.2.6-9]. A similar result was also proven in [DS00,

Theorem 3.1]. However, we need the deformation to be smooth, we need to work in Whitney cubes rather than in a grid of cubes of the same size, and we also need estimates on the measure of the whole deformation.

To deal with families of dyadic cubes it will be convenient to introduce some more notation. We shall follow [Alm86, 1.1–1.9].

**7.1 Definition.** Let  $k \in \{0, 1, \dots, n\}$  and  $Q = [0, 1]^k \subseteq \mathbf{R}^k$ . We say that  $R \subseteq \mathbf{R}^n$  is a *cube* if there exist  $p \in \mathbf{O}^*(n, k)$  and  $o \in \mathbf{R}^n$  and  $l \in (0, \infty)$  such that  $R = \tau_o \circ p^* \circ \mu_l[Q]$ . We call  $\mathbf{o}(R) = o$  the *corner* of  $R$  and  $\mathbf{l}(R) = l$  the *side-length* of  $R$ . We also set

- $\dim(R) = k$  – the *dimension* of  $R$ ,
- $\mathbf{c}(R) = \mathbf{o}(R) + \frac{1}{2}\mathbf{l}(R)(1, 1, \dots, 1)$  – the *centre* of  $R$ ,
- $\text{Bdry}_c(R) = \tau_{\mathbf{o}(R)} \circ p^* \circ \mu_{\mathbf{l}(R)}[\text{Bdry } Q]$  – the *boundary* of  $R$ ,
- $\text{Int}_c(R) = R \sim \text{Bdry}_c(R)$  – the *interior* of  $R$ .

**7.2 Definition.** Let  $k \in \{0, 1, \dots, n\}$ , and  $N \in \mathbf{Z}$ , and  $Q = [0, 1]^k \subseteq \mathbf{R}^k$ , and  $e_1, \dots, e_n$  be the standard basis of  $\mathbf{R}^n$ , and  $f_1, \dots, f_k$  be the standard basis of  $\mathbf{R}^k$ .

We define  $\mathbf{K}_k(N)$  to be the set of all cubes  $R \subseteq \mathbf{R}^n$  of the form  $R = \tau_v \circ p^* \circ \mu_{2^{-N}}[Q]$ , where  $v \in \mu_{2^{-N}}[\mathbf{Z}^n]$  and  $p \in \mathbf{O}^*(n, k)$  is such that  $p^*(f_i) \in \{e_1, \dots, e_n\}$  for  $i = 1, 2, \dots, k$ .

We also set

$$\mathbf{K}_k = \bigcup \{ \mathbf{K}_k(N) : N \in \mathbf{Z} \}, \quad \mathbf{K} = \mathbf{K}_n, \quad \mathbf{K}_* = \bigcup \{ \mathbf{K}_k : k \in \{0, 1, \dots, n\} \}.$$

**7.3 Definition.** Let  $k \in \{0, 1, \dots, n\}$ ,  $N \in \mathbf{Z}$ , and  $K \in \mathbf{K}_k(N)$ . We say that  $L \in \mathbf{K}_*$  is a *face* of  $K$  if and only if  $L \subseteq K$  and  $L \in \mathbf{K}_j(N)$  for some  $j \in \{0, 1, \dots, k\}$ .

**7.4 Definition** (cf. [Alm86, 1.5]). Let  $\mathcal{F} \subseteq \mathbf{K}$ . We say that  $\mathcal{F}$  is *admissible* if

- (a) if  $K, L \in \mathcal{F}$  and  $K \neq L$ , then  $\text{Int}_c(K) \cap \text{Int}_c(L) = \emptyset$ ,
- (b) if  $K, L \in \mathcal{F}$  and  $K \cap L \neq \emptyset$ , then  $\frac{1}{2} \leq \mathbf{l}(L)/\mathbf{l}(K) \leq 2$ ,
- (c) if  $K \in \mathcal{F}$ , then  $\text{Bdry}_c(K) \subseteq \bigcup \{ L \in \mathcal{F} : L \neq K \}$ .

**7.5 Definition** (cf. [Alm86, 1.6]). Let  $U \subseteq \mathbf{R}^n$  be an open set and  $e_1, \dots, e_n$  be the standard basis of  $\mathbf{R}^n$ . We define the *Whitney family*  $\mathbf{WF}(U)$  corresponding to  $U$  to consist of those cubes  $K \in \mathbf{K}$  for which

- $\text{dist}_\infty(K, \mathbf{R}^n \sim U) > 2\mathbf{l}(K)$ ,
- if  $K \subseteq L \in \mathbf{K}$  and  $\mathbf{l}(L) = 2\mathbf{l}(K)$ , then  $\text{dist}_\infty(L, \mathbf{R}^n \sim U) \leq 4\mathbf{l}(K)$ .

where  $\text{dist}_\infty(x, y) = \max\{|(x - y) \bullet e_i| : i = 1, 2, \dots, n\}$  for  $x, y \in \mathbf{R}^n$  and  $\text{dist}_\infty(A, B) = \inf\{\text{dist}_\infty(x, y) : x \in A, y \in B\}$  for  $A, B \subseteq \mathbf{R}^n$ .

**7.6 Remark.** If  $U \subseteq \mathbf{R}^n$  is open, then the Whitney family  $\mathbf{WF}(U)$  is admissible.

**7.7 Definition** (cf. [Alm86, 1.8]). Let  $\mathcal{F} \subseteq \mathbf{K}$  be admissible. We define the *cubical complex*  $\mathbf{CX}(\mathcal{F})$  of  $\mathcal{F}$  to consist of all those cubes  $K \in \mathbf{K}_*$  for which

- $K$  is a face of some cube from  $\mathcal{F}$ ,
- if  $\dim(K) > 0$ , then  $\mathbf{l}(K) \leq \mathbf{l}(L)$  whenever  $L$  is a face of some cube in  $\mathcal{F}$  with  $\dim(K) = \dim(L)$  and  $\text{Int}_c(K) \cap \text{Int}_c(L) \neq \emptyset$ .

**7.8 Remark.** The second item of 7.7 means that whenever two top dimensional cubes  $P$  and  $Q$  from  $\mathcal{F}$  touch and  $P$  is smaller than  $Q$  and  $F$  is a lower dimensional face of  $Q$  such that  $Q \cap P \subseteq F$ , then the cubical complex  $\text{CX}(\mathcal{F})$  does not contain  $F$  but rather cubes coming from subdivision of  $F$ . This is a key property allowing to construct deformations onto skeletons of  $\text{CX}(\mathcal{F})$ .

Now we are ready to construct a map which is the main building block for the deformation theorem 7.13. Given  $a \in (-1, 1)^n$  we need to construct a  $\mathcal{C}^\infty$  smooth function  $\mathbf{R}^n \rightarrow \mathbf{R}^n$  which maps the pointed cube  $Q \sim \{a\} = [-1, 1]^n \sim \{a\}$  onto  $\text{Bdry } Q$ , preserves all the lower dimensional skeletons of  $Q$ , preserves the neighbouring dyadic cubes of side length 1, is very close to the identity on  $\text{Bdry } Q$ , and is the identity outside a small neighbourhood of  $Q$ . Moreover, if  $\text{dist}(a, \text{Bdry } Q) \geq 1/2$ , then we need to control the derivative at  $x \in Q \sim \{a\}$  of this function by a quantity of magnitude  $|x - a|^{-1}$ . To achieve all this we proceed as follows. First we apply a diffeomorphism of  $\mathbf{R}^n$  which preserves  $Q$  and moves  $a$  onto the origin – this step is necessary to preserve the neighbouring cubes. Then, we use the “central projection” constructed in 6.5 to map  $Q \sim \{a\}$  onto the boundary of some convex set  $V$  with smooth boundary such that  $Q \subseteq V$ . Finally, we employ the “smooth retraction” produced in 5.4 to map  $\text{Bdry } V$  onto  $\text{Bdry } Q$ .

**7.9 Lemma.** *Let  $Q = [-1, 1]^n$  and  $\varepsilon \in (0, \frac{1}{4})$ . There exists  $\Gamma = \Gamma(n) \in (0, \infty)$  such that for each  $a \in \text{Int}(Q)$  there exists  $\varphi_{a,\varepsilon} : \mathbf{R}^n \sim \{a\} \rightarrow \mathbf{R}^n$  of class  $\mathcal{C}^\infty$  such that*

- (a)  $\varphi_{a,\varepsilon}(x) = x$  for  $x \in \mathbf{R}^n$  with  $\text{dist}(x, Q) \geq \varepsilon$ .
- (b)  $|\varphi_{a,\varepsilon}(x) - x| \leq \varepsilon$  for  $x \in \mathbf{R}^n \sim Q$ .
- (c)  $\varphi_{a,\varepsilon}[Q \sim \{a\}] = \text{Bdry } Q$ .
- (d) If  $\kappa \in \{-1, 0, 1\}^n \sim \{(0, 0, \dots, 0)\}$  and  $\lambda \in \{-2, -1, 1, 2\}^n \sim \{-2, 2\}^n$  and  $C_\kappa, F_\kappa, T_\kappa, c_\kappa, R_\lambda$  are defined as in 5.1, then

$$\begin{aligned} \varphi_{a,\varepsilon}[F_\kappa] &\subseteq \text{Clos } F_\kappa, & \varphi_{a,\varepsilon}[C_\kappa] &\subseteq \text{Clos } C_\kappa, & \varphi_{a,\varepsilon}[c_\kappa + T_\kappa] &\subseteq c_\kappa + T_\kappa, \\ \varphi_{a,\varepsilon}[R_\lambda] &\subseteq R_\lambda, & \varphi_{a,\varepsilon}[T_\kappa \sim Q] &\subseteq T_\kappa, & a \in T_\kappa &\text{ implies } \varphi_{a,\varepsilon}[T_\kappa \sim \{a\}] \subseteq T_\kappa. \end{aligned}$$

- (e) For  $x \in \text{Int } Q \sim \{a\}$  we have

$$\|\text{D}\varphi_{a,\varepsilon}(x)\| \leq \frac{\Gamma}{2|x - a| \text{dist}(a, \mathbf{R}^n \sim Q)}.$$

- (f) For  $x \in \mathbf{R}^n$  with  $\text{dist}(x, \text{Bdry } Q) \leq \min\{\frac{1}{2} \text{dist}(a, \mathbf{R}^n \sim Q), \frac{1}{4}\}$  there holds

$$\|\text{D}\varphi_{a,\varepsilon}(x)\| \leq \Gamma.$$

- (g) For  $x \in \mathbf{R}^n$  with  $\text{dist}(x, \text{Bdry } Q) \leq \min\{\frac{1}{2} \text{dist}(a, \mathbf{R}^n \sim Q), \frac{1}{4}\}$  we obtain

$$|\varphi_{a,\varepsilon}(x) - x| \leq \Gamma(\text{dist}(x, \text{Bdry } Q) + \varepsilon).$$

- (h) If  $x \in \mathbf{R}^n$ ,  $\delta \in \mathbf{R}$ ,  $0 < \delta < \min\{\frac{1}{2} \text{dist}(a, \mathbf{R}^n \sim Q), \frac{1}{4}\}$ , and  $\text{dist}(x, \text{Bdry } Q) \leq \delta$ , then

$$\text{conv}\{x, \varphi_{a,\varepsilon}(x)\} \subseteq \text{Bdry } Q + \mathbf{B}(0, \delta).$$

*In particular  $\text{dist}(\varphi_{a,\varepsilon}(x), \text{Bdry } Q) \leq \text{dist}(x, \text{Bdry } Q)$  for each  $x \in \mathbf{R}^n \sim \{a\}$ .*

(i) Let  $X \subseteq \text{Bdry } Q$  be compact and convex,  $\kappa \in \{-1, 0, 1\}^n$ ,  $T_\kappa$  be defined as in 5.1. For  $a \in \text{Int } Q$  define  $E_a = \varphi_{a,\varepsilon}^{-1}[X]$ . Then

$$(27) \quad \lim_{b \rightarrow a} d_{\mathcal{H}}(E_a, E_b) = 0 \quad \text{for } a \in \text{Int } Q.$$

Moreover, if  $\mu$  is a Radon measure over  $\mathbf{R}^n$  and  $\dim T_\kappa = k$ , then for  $\mathcal{H}^k$  almost all  $a \in (-\frac{1}{2}, \frac{1}{2})^n \cap T_\kappa$

$$\mu\left(\bigcap_{\delta>0} \bigcup_{b \in \mathbf{B}(a,\delta) \cap T_\kappa} ((E_a \sim E_b) \cup (E_b \sim E_a)) \cap T_\kappa\right) = 0.$$

*Proof.* For each  $\delta \in [0, \infty)$  let  $p_\delta, t_\delta$  define the central projection onto  $\text{Bdry}(Q + \mathbf{U}(0, \delta))$  as in 6.2. Observe that

- $t_\delta$  is continuous for each  $\delta \in [0, \infty)$ ,
- $t_\delta(x) \leq t_\sigma(x)$  whenever  $x \in \mathbf{R}^n \sim \{0\}$  and  $0 \leq \delta \leq \sigma < \infty$ ,
- $\lim_{\delta \downarrow 0} t_\delta(x) = t_0(x)$  for  $x \in \mathbf{R}^n \sim \{0\}$ .

Employing the Dini Theorem (cf. [Rud76, 7.13]), we see that  $t_\delta$  converge uniformly to  $t_0$  as  $\delta \downarrow 0$  on compact subsets of  $\mathbf{R}^n \sim \{0\}$ . Therefore, there exists  $\delta_0 \in (0, \varepsilon)$  such that

$$(28) \quad \text{Bdry } Q \subseteq \{x \in \mathbf{R}^n : 1 < t_\delta(x) < 1 + 2^{-10} n^{-1/2} \varepsilon\} \quad \text{for } \delta \in (0, \delta_0].$$

Set  $\iota = \min\{\delta_0, 2^{-10} \varepsilon / \Gamma_{5.4}\}$ . Choose an open convex set  $V \subseteq \mathbf{R}^n$  with  $\mathcal{C}^\infty$  smooth boundary and such that  $Q + \mathbf{B}(0, \iota/4) \subseteq V \subseteq Q + \mathbf{B}(0, \iota)$ . Such  $V$  is easily constructed, e.g., by taking first the set  $\tilde{V} = Q + \mathbf{B}(0, \iota/2)$ , representing  $\text{Bdry } \tilde{V}$  locally as (rotated) graph of some convex function, and then mollifying this function. Employ 5.4 with  $2^{-10} \varepsilon$  in place of  $\varepsilon$  to obtain the map  $l \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R}^n)$ . Apply 6.5 with  $2^{-10} \iota$ ,  $V$  in place of  $\varepsilon$ ,  $V$  to construct the map  $q \in \mathcal{C}^\infty(\mathbf{R}^n \sim \{0\}, \mathbf{R}^n)$ .

Fix a symmetric non-negative mollifier  $\phi \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$  whose support equals  $[-1, 1]$  and set  $\phi_\rho(s) = \rho^{-1} \phi(s/\rho)$  for  $s \in \mathbf{R}$  and  $\rho > 0$ . Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbf{R}^n$ . For  $i = 1, 2, \dots, n$  set  $\rho_i = \min\{\frac{1}{2}, 1 - |a \bullet e_i|\}$  and let  $\bar{f}_{a,i} : \mathbf{R} \rightarrow \mathbf{R}$  be the continuous piece-wise affine map satisfying

$$\begin{aligned} \bar{f}_{a,i}(t) &= t \quad \text{if } t \geq 1 - \frac{5}{8}\rho_i \text{ or } t \leq -1 + \frac{5}{8}\rho_i, \\ \bar{f}_{a,i}(t) &= t - a \bullet e_i \quad \text{if } |t - a \bullet e_i| \leq \frac{1}{8}\rho_i, \\ \bar{f}_{a,i} &\text{ is affine on } [-1 + \frac{5}{8}\rho_i, a \bullet e_i - \frac{1}{8}\rho_i] \text{ and on } [a \bullet e_i + \frac{1}{8}\rho_i, 1 - \frac{5}{8}\rho_i]. \end{aligned}$$

Next, define  $f_{a,i} = \bar{f}_{a,i} * \phi_{\rho_i/8} \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ . We obtain

- if  $a \bullet e_i = 0$ , then  $f_{a,i}(t) = t$  for  $t \in \mathbf{R}$ ;
- if  $|a \bullet e_i| > 0$ , then

$$\begin{aligned} f_{a,i}(a \bullet e_i) &= 0, \quad f_{a,i}(t) = t \quad \text{for } t \in \mathbf{R} \text{ with } |t| \geq 1 - \frac{1}{2}\rho_i, \\ \frac{1}{2(2 - \rho_i)} &< f'_{a,i}(t) < \frac{2}{\rho} \quad \text{and} \quad |f_i(t)| \geq \frac{1}{2}|t - a \bullet e_i| \quad \text{for } t \in \mathbf{R}. \end{aligned}$$

Define the map  $f_a : \mathbf{R}^n \rightarrow \mathbf{R}^n$  by  $f_a(x) = \sum_{i=1}^n f_{a,i}(x \bullet e_i) e_i$ . Then

$$(29) \quad \begin{aligned} f_a(a) &= 0, \quad |f_a(x)| \geq \frac{1}{2}|x - a|, \quad f_a \text{ is a diffeomorphism of class } \mathcal{C}^\infty, \\ f_a(x) &= x \quad \text{for } x \in \mathbf{R}^n \text{ with } 2 \text{dist}(x, \mathbf{R}^n \sim Q) \leq \text{dist}(a, \mathbf{R}^n \sim Q), \\ \text{and} \quad \frac{1}{2(2 - \text{dist}(a, \mathbf{R}^n \sim Q))} &\leq \|Df_a(x)\| \leq \frac{2}{\text{dist}(a, \mathbf{R}^n \sim Q)} \quad \text{for } x \in Q. \end{aligned}$$



Define  $\varphi_{a,\varepsilon} = l \circ q \circ f_a$ . Clearly  $\varphi_{a,\varepsilon} : \mathbf{R}^n \sim \{a\} \rightarrow \mathbf{R}^n$  is of class  $\mathcal{C}^\infty$ .

*Proof of (a):* If  $x \in \mathbf{R}^n$  satisfies  $\text{dist}(x, Q) \geq \varepsilon$ , then  $x \in \mathbf{R}^n \sim V$ , so  $q(f_a(x)) = q(x) = x$  and  $\varphi_{a,\varepsilon}(x) = l(x) = x$ .

*Proof of (b):* Let  $x \in \mathbf{R}^n \sim Q$ . Then  $f_a(x) = x$ . If  $x \in \mathbf{R}^n \sim V$ , then  $q(x) = x$  and  $|\varphi_{a,\varepsilon}(x) - x| = |l(x) - x| \leq 2^{-10}\varepsilon$  by 5.4(f). Let  $p, t$  define the central projection onto  $\text{Bdry } V$  as in 6.2. If  $x \in V \sim Q$ , then by 6.5(e) and (28) we have

$$|q(x) - x| \leq |p(x) - x| = (t(x) - 1)|x| \leq 2^{-10}n^{-1/2}\varepsilon 2\sqrt{n} \leq 2^{-9}\varepsilon.$$

Hence,  $|\varphi_{a,\varepsilon}(x) - x| \leq |l(q(x)) - q(x)| + |q(x) - x| \leq \varepsilon$  by 5.4(f).

*Proof of (c):* If  $x \in Q \sim \{a\}$ , then  $f_a(x) \in Q \sim \{0\}$  and  $\text{dist}(f_a(x), \mathbf{R}^n \sim V) \geq \iota/4$ , so  $q(f_a(x)) \in \text{Bdry } V$  by 6.5(b). Consequently  $\text{dist}(q(f_a(x)), Q) \leq \iota \leq 2^{-10}\varepsilon/\Gamma_{5.4}$ , which implies  $\varphi_{a,\varepsilon}(x) = l(q(f_a(x))) \in \text{Bdry } V$  by 5.4(b).

*Proof of (d):* For  $\kappa \in \{-1, 0, 1\}^n$  let  $F_\kappa, C_\kappa, T_\kappa$ , and  $c_\kappa$  be defined as in 5.1. Fix a  $\kappa \in \{-1, 0, 1\}^n$  with  $\kappa \neq (0, \dots, 0)$ . Let  $x \in F_\kappa$  and note that  $f_a(x) = x$ . Recall  $\bigcup\{C_\lambda : \lambda \in \{-1, 0, 1\}^n \sim \{(0, \dots, 0)\}\} = \mathbf{R}^n \sim \text{Int } Q$ , so there exists  $\lambda \in \{-1, 0, 1\}^n$  such that  $q(x) \in C_\lambda$ . Since  $q(x) = tx$  for some  $t > 1$ , it follows from the definition of  $C_\lambda$  and  $C_\kappa$  that  $\lambda_j = \kappa_j$  whenever  $\kappa_j \neq 0$ , which implies that  $F_\lambda \subseteq \text{Clos } F_\kappa$ . Since  $l[F_\lambda] \subseteq F_\lambda$  we get  $\varphi_{a,\varepsilon}(x) = l(q(x)) \in \text{Clos}(F_\kappa)$ . Similarly, if  $x \in C_\kappa$ , then  $f_a(x) = x$  and there exists  $\lambda \in \{-1, 0, 1\}^n$  such that  $q(x) \in C_\lambda$  and  $q(x) = tx$  for some  $t \geq 1$ ; hence,  $C_\lambda \subseteq \text{Clos } C_\kappa$  and  $\varphi_{a,\varepsilon}(x) = l(q(x)) \in \text{Clos}(C_\kappa)$  because  $l[C_\lambda] \subseteq C_\lambda$ .

For  $y \in (c_\kappa + T_\kappa) \sim V$  we have  $q(f_a(y)) = q(y) = y$  and then  $\varphi_{a,\varepsilon}(y) = l(y) \in c_\kappa + T_\kappa$  by 5.4(b). If  $y \in (c_\kappa + T_\kappa) \cap V$ , then  $q(f_a(y)) = q(y) = ty$  for some  $t \geq 1$ , by 6.5(d), and, as before,  $q(y) \in C_\lambda$  for some  $\lambda \in \{-1, 0, 1\}^n$  such that  $\lambda_j = \kappa_j$  whenever  $\kappa_j \neq 0$ . In this case  $\text{dist}(y, Q) \leq \varepsilon/\Gamma_{5.4}$  and we can apply 5.4(b) to see that  $l(q(y)) \in F_\lambda \subseteq \text{Clos } F_\kappa \subseteq c_\kappa + T_\kappa$ .

Assume now  $\lambda \in \{-2, -1, 1, 2\}^n \sim \{-2, 2\}^n$  and  $x \in R_\lambda$ . If  $x \notin V$ , then  $\varphi_{a,\varepsilon}(x) = l(x) \in R_\lambda$  by 5.4(b). Thus, assume  $x \in R_\lambda \cap V$ . Let  $\kappa \in \{-1, 0, 1\}^n$  be such that  $\kappa_i = \lambda_i$  if  $|\lambda_i| = 1$  and  $\kappa_i = 0$  if  $|\lambda_i| = 2$  for  $i = 1, 2, \dots, n$ . We already know that  $l(q(x)) \in \text{Clos } F_\kappa$  so it suffices to show that  $(q(x) \bullet e_j)\lambda_j \geq 0$  whenever  $|\lambda_j| = 2$  for some  $j \in \{1, \dots, n\}$  but this is clear since  $q(x) = tx$  for some  $t > 0$ .

If  $a \in T_\kappa$ , then  $f_a[T_\kappa \sim \{a\}] \subseteq T_\kappa \sim \{0\}$ . For  $x \in T_\kappa \sim \{0\}$  we have  $q(x) = tx$  for some  $t \in [1, \infty)$ , so  $q[T_\kappa \sim \{0\}] \subseteq T_\kappa$ . Finally,  $l[T_\kappa \sim \{0\}] \subseteq T_\kappa$ ; hence,  $\varphi_{a,\varepsilon}[T_\kappa \sim \{0\}] \subseteq T_\kappa$ . If  $x \in T_\kappa \sim Q$ , then  $f_a(x) = x$  so  $\varphi_{a,\varepsilon}[T_\kappa \sim Q] \subseteq T_\kappa$  as before.

*Proof of (e):* Set  $\Delta = \inf\{\nu(y) \bullet \frac{y}{|y|} : y \in \text{Bdry } V\}^{-1}$ . Note that  $\Delta$  can be bounded by a constant depending only on  $n$  and, in particular, independently of  $\varepsilon$ . Employ 6.5(f) to get for  $z \in V \sim \{0\}$

$$(30) \quad \|Dq(z)\| \leq \frac{5\Delta \sup\{|y| : y \in \text{Bdry } V\}}{|z|} \leq \frac{10\Delta}{|z|}.$$

Hence, combining 5.4(e), (29), and (30), for  $x \in Q \sim \{a\}$

$$\begin{aligned} \|D\varphi_{a,\varepsilon}(x)\| &\leq \|Dl(q \circ f_a(x))\| \cdot \|Dq(f_a(x))\| \cdot \|Df_a(x)\| \\ &\leq \frac{10\Gamma_{5.4}\Delta}{|f_a(x)| \text{dist}(a, \mathbf{R}^n \sim Q)} \leq \frac{20\Gamma_{5.4}\Delta}{|x - a| \text{dist}(a, \mathbf{R}^n \sim Q)}. \end{aligned}$$

*Proof of (f):* If  $z \in \mathbf{R}^n \sim V$ , then  $\|Dq(z)\| = 1$  and  $|z| > 1$ . If  $z \in V$  and  $\text{dist}(z, \text{Bdry } Q) \leq \frac{1}{4}$ , then  $|z| \geq \frac{3}{4}$ , so  $\|Dq(z)\| \leq 15\Delta$ . Altogether, for  $x \in \mathbf{R}^n$  satisfying

$$\text{dist}(x, \text{Bdry } Q) \leq \min\{\frac{1}{2} \text{dist}(a, \mathbf{R}^n \sim Q), \frac{1}{4}\}$$

we have  $f_a(x) = x$  and, by 5.4(e),

$$\|D\varphi_{a,\varepsilon}(x)\| \leq 15\Delta\Gamma_{5.4}.$$

*Proof of (g):* Let  $x \in \mathbf{R}^n$  satisfy  $\text{dist}(x, \text{Bdry } Q) \leq \min\{\frac{1}{2} \text{dist}(a, \mathbf{R}^n \sim Q), \frac{1}{4}\}$ . Let  $y \in \mathbf{R}^n$  be such that  $\text{dist}(y, Q) = \varepsilon$  and  $|x - y| \leq \text{dist}(x, \text{Bdry } Q) + \varepsilon$ . Then,  $\|D\varphi_{a,\varepsilon}(tx + (1-t)y)\| \leq 15\Delta\Gamma_{5.4}$  for each  $t \in [0, 1]$  and  $\varphi_{a,\varepsilon}(y) = y$ , so

$$|\varphi_{a,\varepsilon}(x) - x| \leq |\varphi_{a,\varepsilon}(x) - \varphi_{a,\varepsilon}(y)| + |y - x| \leq (15\Delta\Gamma_{5.4} + 1)(\text{dist}(x, \text{Bdry } Q) + \varepsilon).$$

*Proof of (h):* Let  $x \in \mathbf{R}^n$ ,  $\delta \in \mathbf{R}$ ,  $0 < \delta < \min\{\frac{1}{2} \text{dist}(a, \mathbf{R}^n \sim Q), \frac{1}{4}\}$ , and  $\text{dist}(x, \text{Bdry } Q) \leq \delta$ ; then  $f_a(x) = x$ .

In case  $x \in \mathbf{R}^n \sim Q$ , if  $\kappa \in \{-1, 0, 1\}^n$  is such that  $x \in C_\kappa$ , then  $\text{dist}(x, Q) = \text{dist}(x, F_\kappa)$  and  $\varphi_{a,\varepsilon}(x) \in \text{Clos } C_\kappa$ . Since  $\text{Clos } C_\kappa \cap (\text{Clos } F_\kappa + \mathbf{B}(0, \delta)) \subseteq \text{Bdry } Q + \mathbf{B}(0, \delta)$  is convex and contains both  $x$  and  $\varphi_{a,\varepsilon}(x)$ , we see that  $\text{conv}\{x, \varphi_{a,\varepsilon}(x)\} \subseteq \text{Bdry } Q + \mathbf{B}(0, \delta)$ .

Assume now  $x \in Q$ . Observe that there exists  $\kappa \in \{-1, 0, 1\}^n$  such that  $\varphi_{a,\varepsilon}(x) \in \text{Clos } F_\kappa$  and  $\text{dist}(x, \text{Bdry } Q) = \text{dist}(x, F_\kappa)$  — this is because  $q$  acts on  $x$  as central projection with centre at the origin. As before, we see that  $x$  and  $\varphi_{a,\varepsilon}(x)$  both lie in the convex set  $\text{Clos } C_\kappa \cap (\text{Clos } F_\kappa + \mathbf{B}(0, \delta))$  so  $\text{conv}\{x, \varphi_{a,\varepsilon}(x)\} \subseteq \text{Bdry } Q + \mathbf{B}(0, \delta)$ .

*Proof of (i):* Set  $Y = (l \circ q)^{-1}[X]$ ,  $P = (-1/2, 1/2)^n$ ,  $R = (-3/4, 3/4)^n$ . Observe that  $E_a = \varphi_{a,\varepsilon}^{-1}[X] = f_a^{-1}[Y]$  and recall that  $f_a$  is a diffeomorphism for each  $a \in \text{Int } Q$ . It follows from the construction that for any compact set  $K \in \text{Int } Q$

$$\lim_{\delta \rightarrow 0} \sup\{|f_a^{-1}(x) - f_b^{-1}(x)| : x \in \mathbf{R}^n, a, b \in K, |a - b| < \delta\} = 0;$$

thus,  $\lim_{b \rightarrow a} d_{\mathcal{H}}(f_a^{-1}[Y], f_b^{-1}[Y]) = 0.$

For  $a \in \text{Int } Q$  let  $B_a$  be the topological boundary of  $E_a \cap T_\kappa$  relative to  $T_\kappa$ . Let  $a \in P \cap T_\kappa$ . It follows from (27) and the construction that

$$\bigcap_{\delta > 0} \bigcup_{b \in \mathbf{B}(a, \delta) \cap T_\kappa} ((E_a \sim E_b) \cup (E_b \cap E_a)) \cap T_\kappa \subseteq R \cap T_\kappa \cap B_a.$$

Without loss of generality we may assume  $T_\kappa = \text{span}\{e_1, \dots, e_k\}$ . Recall the construction of the maps  $l$  and  $q$  to see that  $Y$  is a convex conical cap over  $l^{-1}[X]$  with vertex at the origin. Let  $B$  be the topological boundary of  $Y \cap T_\kappa$  relative to  $T_\kappa$ . Define affine lines  $L_{a,i} = \{a + te_i : t \in \mathbf{R}\}$  for  $a \in \mathbf{R}^n$  and  $i \in \{1, 2, \dots, n\}$ . Since  $B$  is the boundary of a convex set and has empty interior in  $T_\kappa$  we see that there exists  $i \in \{1, 2, \dots, k\}$  such that  $B \cap L_{a,i}$  contains at most two points for each  $a \in T_\kappa$  and we may decompose  $B$  into two disjoint sets  $B = B_1 \cup B_2$  so that for each  $a \in T_\kappa$  if  $a + t_1 e_i \in B_1 \cap L_{i,a}$  and  $a + t_2 e_i \in B_2 \cap L_{i,a}$ , then  $t_1 < t_2$ . Define  $B_{1,a} = f_a^{-1}[B_1]$  and  $B_{2,a} = f_a^{-1}[B_2]$  for  $a \in T_\kappa$ . Then  $B_a$  equals the disjoint sum of  $B_{1,a}$  and  $B_{2,a}$  for each  $a \in \text{Int } Q \cap T_\kappa$  because  $f_a$  is a diffeomorphism. Clearly, it suffices to show that if  $j \in \{1, 2\}$ , then for  $\mathcal{H}^k$  almost all  $a \in P \cap T_\kappa$  we have

$$\mu(R \cap T_\kappa \cap B_{j,a}) = 0.$$

Fix  $j \in \{1, 2\}$ . Observe that if  $a \in P \cap T_\kappa$  and  $t \in (-3/4, 3/4)$ , then the map  $g_{a,t} : L_{a,i} \cap P \rightarrow \mathbf{R}$  given by  $g_{a,t}(b) = f_{b,i}^{-1}(t)$  is strictly increasing. In consequence, if  $b, c \in L_{a,i} \cap P$  and  $b \neq c$ , then  $R \cap T_\kappa \cap B_{j,b} \cap B_{j,c} = \emptyset$ . Since  $\mu$  is Radon, there exists at most countably many  $b \in L_{a,i} \cap P$  for which  $\mu(R \cap T_\kappa \cap B_{j,b}) > 0$ . In particular, we obtain for each  $a \in P \cap T_\kappa$

$$\mathcal{H}^1(\{b \in L_{a,i} \cap P : \mu(R \cap T_\kappa \cap B_{j,b}) > 0\}) = 0.$$

Since  $\mathcal{H}^k \llcorner T_\kappa$  coincides with the Lebesgue measure  $\mathcal{L}^k$  on  $T_\kappa$  and  $\mathcal{L}^k$  is the product of  $k$  copies of the one dimensional Lebesgue measure (cf. [Fed69, 2.6.5]), we may use the Fubini Theorem [Fed69, 2.6.2(3)] to conclude the proof.  $\square$

The next lemma is a counterpart of [Fed69, 4.2.7]. Given arbitrary Radon measures  $\mu_1, \dots, \mu_l$ , and numbers  $m_1, \dots, m_l$ , and a  $k$ -plane  $T_\kappa$  with  $\max\{m_1, \dots, m_l\} < k \leq n$  we prove that there are enough good points  $a \in [-1/2, 1/2]^n \cap T_\kappa$  for which the integral  $\int_Q \|D\varphi_{a,\varepsilon}\|^{m_i} d\mu_i$  is controlled by  $\mu_i(Q)$ . Later we shall apply this lemma to measures  $\mu$  defined as the restriction of  $\mathcal{H}^m$  to some  $m$  dimensional set  $\Sigma \subseteq \mathbf{R}^n$  with density.

**7.10 Lemma.** *Suppose*

$$\begin{aligned} k, N \in \mathcal{P}, \quad k \leq n, \quad \kappa \in \{-1, 0, 1\}^n \text{ is such that } \mathcal{H}^0(\{j : \kappa_j = 0\}) = k, \\ Q \text{ and } T_\kappa \text{ are as in 5.1,} \quad \varepsilon \in (0, \tfrac{1}{4}), \quad A = T_\kappa \cap [-\tfrac{1}{2}, \tfrac{1}{2}]^n, \\ m_1, \dots, m_l \in (0, k), \quad \mu_1, \dots, \mu_l \text{ are Radon measures over } \mathbf{R}^n. \end{aligned}$$

For  $a \in \text{Int } Q$  let  $\varphi_{a,\varepsilon} : \mathbf{R}^n \sim \{a\} \rightarrow \mathbf{R}^n$  be the map constructed in 7.9 and set

$$\begin{aligned} \Gamma(k, m) &= \Gamma_{7.9} \frac{k\alpha(k)}{k-m} k^{(k-m)/2} \quad \text{for } m \in (0, k), \\ E &= \left\{ a \in A : \int_Q \|D\varphi_{a,\varepsilon}\|^{m_i} d\mu_i \leq l\Gamma(k, m_i)\mu_i(Q) \text{ for } i \in \{1, 2, \dots, l\} \right\}. \end{aligned}$$

Then  $\mathcal{L}^k(E) > 0$ .

*Proof.* Employing 7.9(e) we have for  $\varepsilon \in (0, 1/4)$ ,  $m \in (0, k)$ ,  $x \in Q$ , and  $y \in A$  satisfying  $|x - y| = \text{dist}(x, A)$

$$\begin{aligned} \int_A \|D\varphi_{a,\varepsilon}(x)\|^m d\mathcal{L}^k(a) &\leq \Gamma_{7.9} \int_A |x - a|^{-m} d\mathcal{L}^k(a) \leq \Gamma_{7.9} \int_A |y - a|^{-m} d\mathcal{L}^k(a) \\ &\leq \Gamma_{7.9} \int_{T_\kappa \cap \mathbf{B}(0, \sqrt{k})} |y|^{-m} d\mathcal{L}^k(y) = \Gamma_{7.9} \frac{k\alpha(k)}{k-m} k^{(k-m)/2} = \Gamma(k, m). \end{aligned}$$

Thus, for  $i \in \{1, 2, \dots, l\}$ , using the Fubini Theorem [Fed69, 2.6.2], we obtain

$$\int_A \int_Q \|D\varphi_{a,\varepsilon}(x)\|^{m_i} d\mu_i^p(x) d\mathcal{L}^k(a) \leq \Gamma(k, m_i)\mu_i(Q).$$

Now, we argue by contradiction. If  $\mathcal{L}^k(E)$  was zero, then we would have

$$1 = \mathcal{L}^k(A) < \sum_{i=1}^l \frac{1}{l\Gamma(k, m_i)\mu_i(Q)} \int_A \int_Q \|D\varphi_{a,\varepsilon}(x)\|^{m_i} d\mu_i(x) d\mathcal{L}^k(a) \leq \sum_{i=1}^l \frac{1}{l} = 1. \quad \square$$

Now, given a cube  $K \in \mathbf{K}_*$  (of arbitrary dimension and size) and sets  $\Sigma_1, \dots, \Sigma_l$  we combine 7.10 and 7.9 to construct a deformation of  $\mathbf{R}^n$  which maps  $\Sigma_i \cap K$  into  $\text{Bdry}_c(K)$  for each  $i = 1, 2, \dots, l$  and preserves all the super-cubes of  $K$  (i.e. those which contain  $K$ ) as well as all the cubes from  $\mathbf{K}_*$  which do not touch  $\text{Int}_c(K)$  and have side length at least  $\frac{1}{2}l(K)$ . Of course we also control the derivative.

**7.11 Lemma.** *Suppose*

$$\begin{aligned} l \in \mathcal{P}, \quad K \in \mathbf{K}_*, \quad k = \dim(K), \quad m_1, \dots, m_l \in \{1, 2, \dots, k\}, \\ \nu_1, \dots, \nu_l \text{ are Radon measures over } \mathbf{R}^n, \quad \Sigma = \bigcup \{\text{spt } \nu_i : i = 1, 2, \dots, l\}, \\ \text{either } \max\{m_1, \dots, m_l\} \leq k-1 \text{ and } \mathcal{H}^k(\Sigma \cap K) = 0 \\ \text{or } m_1 = \dots = m_l = k \text{ and } \Sigma \cap K \neq K. \end{aligned}$$

*Then for each  $\varepsilon_0 \in (0, \frac{1}{4}\mathbf{l}(K))$  there exist  $\varepsilon \in (0, \varepsilon_0]$ , a neighbourhood  $U$  of  $\Sigma$  in  $\mathbf{R}^n$ , and a map  $\varphi \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R}^n)$  such that*

- (a)  $\varphi \in \mathfrak{D}(K + \mathbf{U}(0, \varepsilon))$ ,
- (b)  $\varphi(x) = x$  for  $x \in \mathbf{R}^n$  satisfying  $\text{dist}(x, K) \geq \varepsilon$ ,
- (c)  $\varphi[U] \cap K = \varphi[U \cap K] \subseteq \text{Bdry}_c(K)$ ,
- (d)  $\varphi[\text{Bdry}_c(K) + \mathbf{B}(0, \varepsilon)] \subseteq \text{Bdry}_c(K) + \mathbf{B}(0, \varepsilon)$ ,
- (e)  $|\varphi(x) - x| \leq \varepsilon$  for  $x \in \mathbf{R}^n$  satisfying  $T_{\mathfrak{t}}(x - \mathbf{c}(K)) \notin T_{\mathfrak{t}}[K]$ , where  $T = \text{Tan}(K, \mathbf{c}(K))$ ,
- (f) if  $L \in \mathbf{K}_*$  satisfies either  $\mathbf{l}(L) \geq \mathbf{l}(K)$  or  $\mathbf{l}(L) \geq \frac{1}{2}\mathbf{l}(K)$  and  $L \cap \text{Int}_c(K) = \emptyset$ , then  $\varphi[L] \subseteq L$ .
- (g)  $\|\text{D}\varphi(x)\| \leq \Gamma$  for  $x \in \mathbf{R}^n$  with  $\text{dist}(x, \text{Bdry}_c(K)) \leq \varepsilon$ ,
- (h) if  $\max\{m_1, \dots, m_l\} \leq k-1$ , then there exists  $\Gamma = \Gamma(k, l) \in (1, \infty)$  such that

$$\int_K \|\text{D}\varphi\|^{m_i} d\nu_i \leq \Gamma \nu_i(K) \quad \text{for } i = 1, 2, \dots, l.$$

*Proof.* Let  $\varepsilon_0 \in (0, \mathbf{l}(K)/4)$ . If  $\max\{m_1, \dots, m_l\} \leq k-1$ , then set  $\varepsilon = \varepsilon_0$ . If  $m_1 = \dots = m_l = k$ , then choose arbitrary  $a_0 \in \text{Int}_c(K) \sim \Sigma$  and set

$$\varepsilon = \min\{\varepsilon_0, 2^{-8} \text{dist}(a_0, \text{Bdry}_c(K))\}.$$

Translating  $\Sigma$  and  $K$  by  $-\mathbf{c}(K)$  we can assume  $\mathbf{c}(K) = 0$ . Set

$$\begin{aligned} T = \text{Tan}(K, \mathbf{c}(K)), \quad \iota = \varepsilon/\sqrt{2}, \quad r = \mu_{2/\mathbf{l}(K)} \circ \tau_{-\mathbf{c}(K)}, \\ \mu_i = (r_{\#} \nu_i) \llcorner T \quad \text{for } i = 1, \dots, l. \end{aligned}$$

Note that  $r[K] = [-1, 1]^n \cap T$ . For  $a \in K$  let  $\varphi_{r(a), 2\iota/\mathbf{l}(K)}$  be the map defined by employing 7.9 with  $r(a)$ ,  $2\iota/\mathbf{l}(K)$  in place of  $a$ ,  $\varepsilon$  and set

$$\psi_a = r^{-1} \circ \varphi_{r(a), 2\iota/\mathbf{l}(K)} \circ r.$$

To choose an appropriate  $a \in K$ , we consider two cases.

- If  $\max\{m_1, \dots, m_l\} \leq k-1$ , then we proceed as follows. Define  $E \subseteq K$  to be the set of all those  $a \in K$  for which  $r(a) \in [-1/2, 1/2]^n$  and

$$(31) \quad \int_{[-1, 1]^n} \|\text{D}\varphi_{r(a), 2\iota/\mathbf{l}(K)}\|^{m_i} d\mu_i \leq l\Gamma_{7.10}(k, m_i)\mu_i([-1, 1]^n) \quad \text{for } i = 1, \dots, l.$$

Apply 7.10 with  $2\iota/\mathbf{l}(K)$  in place of  $\varepsilon$  to conclude that  $\mathcal{L}^k(E) > 0$ . Since we have  $\mathcal{H}^k(\Sigma \cap K) = 0$ , we may choose  $a \in E \sim \Sigma$ .

- If  $m_1 = \dots = m_l = k$ , then we set  $a = a_0$ .

Since  $\Sigma \cap K$  is compact we have

$$d = \frac{1}{2} \min\{\iota, \text{dist}(a, \Sigma)\} > 0.$$

Let  $\alpha \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$  be such that  $\alpha(t) = 1$  for  $t \geq 7/8$ ,  $\alpha(t) = 0$  for  $t \leq 1/4$ , and  $0 < \alpha'(t) < 2$  for  $t \in (0, 1)$ . Recall that we assumed  $c(K) = 0$ ; in particular  $K \subseteq T$ . Set

$$\begin{aligned} \psi(x) &= \begin{cases} x & \text{for } x \in \mathbf{B}(a, d/4) \\ \alpha(|x - a|/d)\psi_a(x) + (1 - \alpha(|x - a|/d))x & \text{for } x \in \mathbf{R}^n \sim \mathbf{B}(a, d/4) \end{cases} \\ \varphi(x) &= T_{\mathfrak{h}}^\perp x + \psi(T_{\mathfrak{h}}x) + (T_{\mathfrak{h}}x - \psi(T_{\mathfrak{h}}x))\alpha(|T_{\mathfrak{h}}^\perp x|/\iota) \quad \text{for } x \in \mathbf{R}^n. \end{aligned}$$

Clearly  $\varphi$  is of class  $\mathcal{C}^\infty$ . Since  $K + \mathbf{U}(0, \varepsilon)$  is convex, we see that (a) is satisfied. Set

$$Q = [-1, 1]^n \quad \text{and} \quad R = r^{-1}[Q] \quad \text{and} \quad U = \Sigma + \mathbf{U}(0, 2^{-3}d).$$

*Proof of (b):* If  $x \in \mathbf{R}^n$  satisfies  $\text{dist}(x, K) \geq \varepsilon$ , then either  $\text{dist}(x, T) \geq \varepsilon/\sqrt{2} \geq \iota$  and then  $\varphi(x) = x$ , or  $\text{dist}(T_{\mathfrak{h}}x, K) \geq \varepsilon/\sqrt{2} \geq \iota$  and then  $\psi_a(x) = x$ , by 7.9(a), and  $\varphi(x) = x$ .

*Proof of (c):* If  $x \in U \cap K$ , then  $\varphi(x) = \psi_a(x)$ . Observe that  $\psi_a(T \sim \{a\}) \subseteq T$  by 7.9(d) because  $a \in T$ . Combining this with  $\psi_a[R \sim \{a\}] \subseteq \text{Bdry } R$ , which holds due to 7.9(c), we see that  $\varphi[U \cap K] \subseteq \text{Bdry}_c(K)$ . Moreover, by 7.9(d) and the definition of  $\varphi$  we have  $\varphi[\mathbf{R}^n \sim K] \subseteq \mathbf{R}^n \sim K$ , so  $\varphi[U] \cap K = \varphi[U \cap K]$ .

*Proof of (d):* Let  $x \in \text{Bdry}_c(K) + \mathbf{B}(0, \varepsilon)$ . Observe that  $T_{\mathfrak{h}}^\perp x = T_{\mathfrak{h}}^\perp \varphi(x)$  because  $\psi_a(T_{\mathfrak{h}}x) \in T$  due to 7.9(d). Moreover,  $\text{Bdry } R \cap (T + \mathbf{B}(0, \varepsilon)) \subseteq \text{Bdry}_c(K) + T^\perp$ , so for any  $z \in \text{Bdry}_c(K) + \mathbf{B}(0, \varepsilon)$  we have  $\text{dist}(z, \text{Bdry}_c(K))^2 = \text{dist}(T_{\mathfrak{h}}z, \text{Bdry } R)^2 + \text{dist}(z, T)^2$ . Noting  $|T_{\mathfrak{h}}^\perp x| \leq \varepsilon$  and using 7.9(d)(h) we can write

$$\begin{aligned} \text{dist}(\varphi(x), \text{Bdry}_c(K))^2 &= \text{dist}(T_{\mathfrak{h}}\varphi(x), \text{Bdry } R)^2 + \text{dist}(\varphi(x), T)^2 \\ &= \text{dist}((1-t)\psi(T_{\mathfrak{h}}x) + tT_{\mathfrak{h}}x, \text{Bdry}_c(K))^2 + |T_{\mathfrak{h}}^\perp x|^2 \\ &\leq \text{dist}(T_{\mathfrak{h}}x, \text{Bdry}_c(K))^2 + |T_{\mathfrak{h}}^\perp x|^2 = \text{dist}(x, \text{Bdry}_c(K))^2, \end{aligned}$$

where  $t = \alpha(|T_{\mathfrak{h}}^\perp x|/\iota)$ . Thus,  $\varphi(x) \in \text{Bdry}_c(K) + \mathbf{B}(0, \varepsilon)$ .

*Proof of (e):* Let  $x \in \mathbf{R}^n$  be such that  $T_{\mathfrak{h}}x \notin K$ . If  $\text{dist}(x, K) \geq \varepsilon$ , then  $\varphi(x) = x$  and there is nothing to prove. Assume  $\text{dist}(x, K) \leq \varepsilon$ . By 7.9(b) we know  $|\psi(T_{\mathfrak{h}}x) - T_{\mathfrak{h}}x| \leq \iota$  so for any  $t \in [0, 1]$  we have  $|(tT_{\mathfrak{h}}x + (1-t)\psi(T_{\mathfrak{h}}x)) - T_{\mathfrak{h}}x| \leq \iota$ . Setting  $t = \alpha(|T_{\mathfrak{h}}^\perp x|/\iota)$  we obtain

$$|\varphi(x) - x| = |T_{\mathfrak{h}}\varphi(x) - T_{\mathfrak{h}}x| \leq |(tT_{\mathfrak{h}}x + (1-t)\psi(T_{\mathfrak{h}}x)) - T_{\mathfrak{h}}x| \leq \iota \leq \varepsilon.$$

*Proof of (f):* Let  $L \in \mathbf{K}_*$  be such that  $\mathbf{l}(L) \geq \frac{1}{2}\mathbf{l}(K)$  and  $L \cap \text{Int}_c(K) = \emptyset$ . Observe that if  $\mathbf{l}(L) > \frac{1}{2}\mathbf{l}(K)$ , then  $L$  is a sum of some cubes from  $\mathbf{K}_*$  which do not intersect  $\text{Int}_c(K)$  and have side length  $\frac{1}{2}\mathbf{l}(K)$ ; thus, it is enough to prove the claim in case  $\mathbf{l}(L) = \frac{1}{2}\mathbf{l}(K)$  – we shall assume this holds. Since  $\varphi(x) = x$  for  $x \in \mathbf{R}^n$  with  $\text{dist}(x, K) \geq \varepsilon$  and  $\varepsilon \leq \frac{1}{4}\mathbf{l}(K)$  we will also assume that  $L \cap K \neq \emptyset$ . For  $\kappa \in \{-1, 0, 1\}^n$  and  $\lambda \in \{-2, -1, 1, 2\}^n$  let  $c_\kappa, T_\kappa, R_\lambda$ , be as in 5.1. If  $\dim(T_{\mathfrak{h}}[L]) = \dim(K)$ , then  $T_{\mathfrak{h}}[L] = T \cap r[R_\lambda]$  for some  $\lambda \in \{-2, -1, 1, 2\}^n$ . If  $\dim(T_{\mathfrak{h}}[L]) < \dim(K)$ , then  $T_{\mathfrak{h}}[L] \subseteq \text{Bdry}_c(K)$  and  $L$  is contained in some face of  $R$ . In this case let  $\kappa \in \{-1, 0, 1\}^n$  be such that  $L \subseteq r[F_\kappa]$  and  $F_\kappa \subseteq F_\sigma$  whenever  $L \subseteq r[F_\sigma]$  for some  $\sigma \in \{-1, 0, 1\}^n$ . If it happens that  $\dim(F_\kappa) > \dim(L)$ , then  $L$  must lie inside  $T_\sigma$  for some

$\sigma \in \{-1, 0, 1\}^n$ . Altogether, there exist  $\lambda \in \{-2, -1, 1, 2\}^n \sim \{-2, 2\}^n$  and  $\kappa, \sigma \in \{-1, 0, 1\}^n$  such that

$$T_{\mathfrak{h}}[L] = r^{-1}[R_\lambda \cap T_\sigma \cap (c_\kappa + T_\kappa) \cap T].$$

Hence,  $\psi[T_{\mathfrak{h}}[L]] \subseteq T_{\mathfrak{h}}[L]$  by 7.9(d). Since  $L$  is convex, we obtain  $\varphi[T_{\mathfrak{h}}[L]] \subseteq T_{\mathfrak{h}}[L]$ . Finally, note that  $L = T_{\mathfrak{h}}[L] + T_{\mathfrak{h}}^\perp[L]$  and  $\varphi[L] = \varphi[T_{\mathfrak{h}}[L]] + T_{\mathfrak{h}}^\perp[L]$ , which proves the claim in case  $L \cap \text{Int}_c(K) = \emptyset$ .

If  $\text{Int}_c(K) \cap L \neq \emptyset$  but  $l(L) \geq l(K)$ , then  $K \subseteq T_{\mathfrak{h}}[L]$ . Clearly  $\varphi[K] \subseteq K$  and  $T_{\mathfrak{h}}[L] \sim K$  is contained in a sum of cubes with side length at least  $\frac{1}{2}l(K)$  which do not intersect  $\text{Int}_c(K)$ . Hence,  $\varphi[T_{\mathfrak{h}}[L]] \subseteq T_{\mathfrak{h}}[L]$  and the claim follows as before.

**Proof of (g):** Assume  $x \in \mathbf{R}^n$  satisfies  $\text{dist}(x, \text{Bdry}_c(K)) \leq \varepsilon$ , then  $\text{dist}(T_{\mathfrak{h}}x, \text{Bdry}_c(K)) \leq \varepsilon$ . Since  $d \leq \iota/2 \leq 2^{-7} \text{dist}(a, \text{Bdry}_c(K))$  we see that  $|T_{\mathfrak{h}}x - a| \geq (1 - 2^{-8}) \text{dist}(a, \text{Bdry}_c(K)) \geq d$  so  $\psi(T_{\mathfrak{h}}x) = \psi_a(T_{\mathfrak{h}}x)$ . Recalling  $\alpha'(t) \leq 2$ ,  $\alpha(t) \leq 1$  for  $t \in \mathbf{R}$ ,  $\iota = \varepsilon/\sqrt{2}$ , and 7.9(f)(g) we get

$$\begin{aligned} \|\text{D}\varphi(x)\| &\leq \|T_{\mathfrak{h}}^\perp\| + \|\text{D}\psi_a(T_{\mathfrak{h}}x) \circ T\| + \|T_{\mathfrak{h}} - \text{D}\psi_a(T_{\mathfrak{h}}x) \circ T\| + 2/\iota |T_{\mathfrak{h}}x - \psi_a(T_{\mathfrak{h}}x)| \\ &\leq 2 + 10\Gamma_{7.9}. \end{aligned}$$

**Proof of (h):** Let us assume  $\max\{m_1, \dots, m_l\} \leq k - 1$ ; hence,  $r(a) \in [-\frac{1}{2}, \frac{1}{2}]^n \cap T$ . Note that for  $i \in \{1, \dots, l\}$  and  $x \in T \sim \mathbf{B}(a, d)$

$$(32) \quad \|\text{D}\varphi(x)\| = \|T_{\mathfrak{h}}^\perp + \text{D}\psi_a(x) \circ T_{\mathfrak{h}}\| \leq 1 + \|\text{D}\varphi_{r(a), 2\iota/l(K)}(r(x))\|$$

Using (32) and the definition of  $\Sigma$ ,  $U$ , and  $\mu_i$  we get

$$(33) \quad \int_K \|\text{D}\varphi\|^{m_i} d\nu_i \leq 2^{m_i-1} \int_{[-1,1]^n} \|\text{D}\varphi_{r(a), 2\iota/l(K)}\|^{m_i} d\mu_i + 2^{m_i-1} \nu_i(K)$$

$$(34) \quad \text{and } \mu_i([-1, 1]^n) = \nu_i(K) \text{ for } i = 1, \dots, l.$$

Combining (31) with (33) and (34) yields (h).  $\square$

Next, we shall prove our main deformation theorem 7.13. Given a finite subset  $\mathcal{A}$  of an admissible family  $\mathcal{F}$  of top dimensional cubes from  $\mathbf{K}$  and some sets  $\Sigma_1, \dots, \Sigma_l$ , we deform all these sets onto the  $m$  dimensional skeleton of  $\mathcal{A}$  using a smooth deformation of  $\mathbf{R}^n$ . Furthermore, we provide estimates on the measure of the deformed sets (i.e. the images of  $\Sigma_i$  for  $i = 1, 2, \dots, l$ ) and, in case  $\Sigma_i$  are rectifiable, also on the measure of the whole deformation (i.e. the images of  $[0, 1] \times \Sigma_i$  for  $i = 1, 2, \dots, l$ ). The basic idea of the proof is simple: we order all the cubes of the cubical complex  $\text{CX}(\mathcal{F})$  which touch the interior of some cube from  $\mathcal{A}$  lexicographically with respect to side length and dimension and then apply 7.11 iteratively to each cube. If the dimensions of  $\Sigma_1, \dots, \Sigma_l$  all equal  $m$ , then we additionally ensure that all the  $m$ -dimensional faces of  $\mathcal{A}$  are either fully covered or not covered at all. During this last step we cannot control the derivative so we actually provide two deformations: one with good estimates (called “ $g$ ”) and one without estimates (called “ $f$ ”) but performing the last step of cleaning the cubes which are not fully covered.

To be able to estimate the measure of the image of  $\Sigma_i$  even if  $\Sigma_i$  is not rectifiable, we need the following simple lemma.

**7.12 Lemma.** *Let  $S \subseteq \mathbf{R}^n$  be such that  $\mathcal{H}^m(S \cap K) < \infty$  for every compact set  $K \subseteq \mathbf{R}^n$ ,  $g \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^N)$  for some  $N \in \mathcal{P}$ , and  $f \in L^1(\mathcal{H}^m \llcorner g[S], \mathbf{R})$  be non-negative. Then*

$$\int f d\mathcal{H}^m \llcorner g[S] \leq \int f \circ g \|Dg\|^m d\mathcal{H}^m \llcorner S.$$

*Proof.* Since  $S$  can be decomposed into a countable sum of compact sets we can assume  $S$  is compact. Furthermore, using standard methods of Lebesgue integration [Fed69, 2.3.3, 2.4.8] we can assume  $f = \mathbb{1}_A$  for some  $\mathcal{H}^m \llcorner g[S]$  measurable set  $A \subseteq \mathbf{R}^N$ . Let  $\varepsilon > 0$ . For each  $x \in S$  choose  $r_x > 0$  so that

$$\|Dg(x)\|^m - \varepsilon \leq \text{Lip}(g|_{\mathbf{B}(x, r_x)})^m \leq \|Dg(x)\|^m + \varepsilon.$$

This is possible since  $Dg$  is continuous. From the family  $\{\mathbf{U}(x, r_x) : x \in S\}$  choose a finite covering  $\mathcal{B} = \{B_1, \dots, B_K\}$  of  $S$ . For  $j = 1, 2, \dots, K$  define  $S_j \subseteq S$ , and  $x_j \in \mathbf{R}^n$ , and  $r_j \in \mathbf{R}$  so that

$$B_j = \mathbf{U}(x_j, r_j) \quad \text{and} \quad S_j = S \cap B_j \sim \bigcup \{B_i : i = 1, 2, \dots, j-1\}.$$

We obtain

$$\begin{aligned} \int_{g[S]} f \, d\mathcal{H}^m &= \mathcal{H}^m(A \cap g[S]) = \mathcal{H}^m(\bigcup \{g[S_j \cap g^{-1}[A]] : j = 1, 2, \dots, K\}) \\ &\leq \sum_{j=1}^K (\|Dg(x_j)\|^m + \varepsilon) \mathcal{H}^m(S_j \cap g^{-1}[A]) = \int_S v_\varepsilon \, d\mathcal{H}^m, \end{aligned}$$

where

$$v_\varepsilon = \sum_{j=1}^K (\|Dg(x_j)\|^m + \varepsilon) \mathbb{1}_{S_j \cap g^{-1}[A]} = f \circ g \sum_{j=1}^K (\|Dg(x_j)\|^m + \varepsilon) \mathbb{1}_{S_j}.$$

We obtain the claim by letting  $\varepsilon \rightarrow 0$  and using the dominated convergence theorem; see [Fed69, 2.4.9].  $\square$

**7.13 Theorem.** *Suppose*

$$\begin{aligned} &\mathcal{F} \subseteq \mathbf{K} \text{ is admissible, } \mathcal{A} \subseteq \mathcal{F} \text{ is finite, } \Sigma_1, \dots, \Sigma_l \subseteq \mathbf{R}^n, \quad m, m_1, \dots, m_l \in \mathcal{P}, \\ &\varepsilon_0 = 2^{-4} \min\{\mathbf{l}(R) : R \in \mathcal{A}\}, \quad \delta = \max\{\mathbf{l}(R) : R \in \mathcal{A}\}, \quad G_{\varepsilon_0} = \bigcup \mathcal{A} + \mathbf{U}(0, \varepsilon_0), \\ &n-1 \geq m = m_1 \geq \dots \geq m_l \geq 1, \quad \Sigma = \bigcup \{\Sigma_i : i = 1, \dots, l\}, \quad \mathcal{H}^{m+1}(\text{Clos } \Sigma \cap G_{\varepsilon_0}) = 0, \\ &\Sigma_i \text{ is } \mathcal{H}^{m_i} \text{ measurable and } \mathcal{H}^{m_i}(\Sigma_i \cap G_{\varepsilon_0}) < \infty \text{ for } i = 1, \dots, l. \end{aligned}$$

*Then for each  $\varepsilon \in (0, \varepsilon_0)$ , setting  $G_\varepsilon = \bigcup \mathcal{A} + \mathbf{U}(0, \varepsilon)$  and  $G_0 = \text{Int } \bigcup \mathcal{A}$ , there exist deformations  $f, g \in \mathcal{C}^\infty(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$  and a neighbourhood  $U$  of  $\Sigma \cap G_\varepsilon$  in  $\mathbf{R}^n$  satisfying:*

- (a)  $f(t, x) = x$  if either  $t = 0$  and  $x \in \mathbf{R}^n$  or  $t \in \mathbf{R}$  and  $x \in \mathbf{R}^n \sim G_\varepsilon$ .
- (b) *There exist  $N, N_0 \in \mathcal{P}$ ,  $N_0 \leq N$ ,  $\varphi_1, \dots, \varphi_N \in \mathfrak{D}(G_\varepsilon)$ , and  $K_1, \dots, K_N \in \mathbf{CX}(\mathcal{F})$  such that for each  $j = 1, \dots, N$  setting  $\psi_0 = \text{id}_{\mathbf{R}^n}$  and  $\psi_j = \varphi_j \circ \psi_{j-1}$  we have*

$$\{x \in \mathbf{R}^n : \varphi_j(x) \neq x\} \subseteq K_j + \mathbf{U}(0, \varepsilon) \quad \text{and} \quad \varphi_j[\psi_{j-1}[U] \cap K_j] \subseteq \text{Bdry}_c(K_j).$$

*Moreover, there exists  $s \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$  such that  $s(0) = 0$ , and  $s(1) = 1$ , and  $0 \leq s'(t) \leq 2$  for  $t \in \mathbf{R}$ , and  $D^k s(0) = 0 = D^k s(1)$  for  $k \in \mathcal{P}$ , and*

$$f(t, \cdot) = s(tN - j)\psi_{j+1} + (1 - s(tN - j))\psi_j = g(tN/N_0, x)$$

*whenever  $j \in \{0, \dots, N-1\}$  and  $t \in \mathbf{R}$  satisfy  $j \leq tN \leq j+1$ .*

- (c)  $f(t, \cdot) \in \mathfrak{D}(G_\varepsilon)$  for each  $t \in I$ .

(d) If  $K \in \mathbf{CX}(\mathcal{F})$ , then  $f(t, \cdot)[K] \subseteq K$  for  $t \in I$ . In particular

$$f(t, \cdot)[\bigcup \mathcal{A}] \subseteq \bigcup \mathcal{A} \quad \text{and} \quad f(t, \cdot)[\mathbf{R}^n \sim \bigcup \mathcal{A}] \subseteq \mathbf{R}^n \sim G_0 \quad \text{for } t \in I.$$

(e)  $g(1, \cdot)[U] \cap G_0 \subseteq \bigcup \{K \in \mathbf{CX}(\mathcal{F}) : K \cap G_0 \neq \emptyset, \dim(K) = m\}$ .

(f) If  $m_1 = \dots = m_l$ , then for each  $K \in \mathbf{CX}(\mathcal{F}) \cap \mathbf{K}_m$  satisfying  $K \cap G_0 \neq \emptyset$  there holds

$$\text{either } \text{Int}_c(K) \cap f(1, \cdot)[\Sigma \cap \bigcup \mathcal{A}] = \emptyset \quad \text{or} \quad K \cap f(1, \cdot)[\Sigma \cap \bigcup \mathcal{A}] = K.$$

(g) There exists  $\Gamma = \Gamma(n, m) \in (0, \infty)$  such that for each  $t \in I$  and  $i \in \{1, \dots, l\}$  and  $Q \in \mathcal{F}$ , setting  $\tilde{Q} = \bigcup \{R \in \mathcal{F} : R \cap Q \neq \emptyset\}$ ,

$$(35) \quad \|Df(t, \cdot)(x)\| < \Gamma \quad \text{for } x \in G_\varepsilon \sim G_0,$$

$$(36) \quad \int_{\Sigma_i \cap Q} \|Dg(t, \cdot)\|^{m_i} d\mathcal{H}^{m_i} < \Gamma \mathcal{H}^{m_i}(\Sigma_i \cap (\tilde{Q} + \mathbf{U}(0, \varepsilon))),$$

$$(37) \quad \mathcal{H}^{m_i}(g(t, \cdot)[\Sigma_i \cap G_\varepsilon]) \stackrel{7.12}{\leq} \int_{\Sigma_i \cap G_\varepsilon} \|Dg(t, \cdot)\|^{m_i} d\mathcal{H}^{m_i} < \Gamma \mathcal{H}^{m_i}(\Sigma_i \cap G_\varepsilon),$$

$$(38) \quad \mathcal{H}^{m_i}(f(1, \cdot)[\Sigma_i] \cap Q) < \Gamma \mathcal{H}^{m_i}(\Sigma_i \cap (\tilde{Q} + \mathbf{U}(0, \varepsilon))),$$

$$(39) \quad \mathcal{H}^{m_i}(f(1, \cdot)[\Sigma_i \cap G_\varepsilon]) < \Gamma \mathcal{H}^{m_i}(\Sigma_i \cap G_\varepsilon);$$

moreover, if  $\Sigma_i$  is  $(\mathcal{H}^{m_i}, m_i)$  rectifiable, then

$$(40) \quad \mathcal{H}^{m_i+1}(g[I \times (\Sigma_i \cap G_\varepsilon)]) \leq \mathcal{H}^{m_i+1}(f[I \times (\Sigma_i \cap G_\varepsilon)]) < \Gamma \delta \mathcal{H}^{m_i}(\Sigma_i \cap G_\varepsilon).$$

*Proof.* Fix  $\varepsilon \in (0, \varepsilon_0)$ . Without loss of generality we assume that  $\Sigma_i \subseteq G_\varepsilon$  for  $i = 1, 2, \dots, l$ . Define

$$\mathcal{C} = \{K \in \mathbf{CX}(\mathcal{F}) : K \cap G_0 \neq \emptyset, \dim(K) \geq m+1\}.$$

Let  $\Delta = \Delta(n, m) \in \mathcal{P}$  be so big that

$$\Delta \geq \max\{\mathcal{H}^0(\{K \in \mathbf{CX}(\mathcal{F}) : K \cap L \neq \emptyset\}) : L \in \mathbf{CX}(\mathcal{F})\}.$$

Let  $\{K_1, \dots, K_{N_0}\} = \mathcal{C}$  be an enumeration of  $\mathcal{C}$  chosen so that for  $1 \leq i \leq j \leq N_0$

$$\text{either } \dim(K_i) = \dim(K_j) \text{ and } \mathbf{l}(K_i) \geq \mathbf{l}(K_j) \quad \text{or} \quad \dim(K_i) \geq \dim(K_j).$$

For each  $j = 0, 1, \dots, N_0$  we shall define inductively maps  $\varphi_j, \psi_j, \zeta_{Q,j}, \eta_{Q,j}$ , measures  $\nu_{Q,j,i}$ , and open sets  $U_j \subseteq \mathbf{R}^n$ , where  $i \in \{1, 2, \dots, l\}$  and  $Q \in \mathcal{F}$ . First we set

$$\psi_0 = \varphi_0 = \eta_{Q,0} = \zeta_{Q,0} = \text{id}_{\mathbf{R}^n}, \quad \nu_{Q,0,i} = \mathcal{H}^{m_i} \llcorner \Sigma_i, \quad U_0 = \mathbf{R}^n.$$

Assume that  $\varphi_{j-1}, \psi_{j-1}, \eta_{Q,j-1}, U_{j-1}, \nu_{j-1,1}, \dots, \nu_{j-1,l}$  are defined for some  $j = 1, 2, \dots, N_0$ . We set

$$\nu_{Q,j,i} = (\eta_{Q,j-1})_\# (\|D\eta_{Q,j-1}\|^{m_i} \mathcal{H}^{m_i} \llcorner \Sigma_i) \quad \text{for } i = 1, 2, \dots, l \text{ and } Q \in \mathcal{F}.$$

Observe, that all the measures  $\nu_{Q,i,j}$  for  $i \in \{1, 2, \dots, l\}$  and  $Q \in \mathcal{F}$  are Radon because  $\eta_{Q,j-1}$  is smooth and proper and  $\mathcal{H}^{m_i} \llcorner \Sigma_i$  is finite. Let  $\varphi_j$  and  $U_j$  be the map and the neighbourhood of  $\Sigma$  constructed by employing 7.11 with

$$K_j, \varepsilon/\Delta, \mathcal{H}^0(\{Q \in \mathcal{F} : Q \cap K_j \neq \emptyset\}), \{(m_i, \nu_{Q,i,j}) : i \in \{1, 2, \dots, l\}, Q \in \mathcal{F}, Q \cap K_j \neq \emptyset\}$$

in place of  $K, \varepsilon_0, l, \{(m_i, \nu_i) : i \in \{1, 2, \dots, l\}\}.$



If  $Q \cap K_j \neq \emptyset$ , we set  $\zeta_{Q,j} = \varphi_j$ , and if  $Q \cap K_j = \emptyset$ , we set  $\zeta_{Q,j} = \text{id}_{\mathbf{R}^n}$ . Next, we define

$$\psi_j = \varphi_j \circ \psi_{j-i}, \quad \eta_{Q,j} = \zeta_{Q,j} \circ \eta_{Q,j-1} \quad \text{for } Q \in \mathcal{F}.$$

If  $m_1 > m_l$ , then we set  $N = N_0$ . If  $m = m_1 = \dots = m_l$ , then we still have to take care of the cubes of dimension  $m$  which are not fully covered. In this case we define

$$\mathcal{C}' = \left\{ K \in \mathbf{CX}(\mathcal{F}) : \begin{array}{l} K \cap G_0 \neq \emptyset, \dim(K) = m, \\ K \cap \psi_{N_0}[\Sigma] \neq K, \text{Int}_c(K) \cap \psi_{N_0}[\Sigma] \neq \emptyset \end{array} \right\}.$$

We enumerate  $\mathcal{C}' = \{K_{N_0+1}, \dots, K_N\}$  so that for  $N_0 < i \leq j \leq N$  we have  $\mathbf{l}(K_i) \geq \mathbf{l}(K_j)$ . For  $j = N_0+1, \dots, N$  we define inductively  $\varphi_j, \psi_j, U_j$  similarly as before by employing 7.11 with  $K_j, \varepsilon/\Delta, l, \mathcal{H}^m \llcorner \psi_{j-1}[\Sigma_1], \dots, \mathcal{H}^m \llcorner \psi_{j-1}[\Sigma_l], m, \dots, m$  in place of  $K, \varepsilon_0, l, \nu_1, \dots, \nu_l, m_1, \dots, m_l$  and we set  $\psi_j = \varphi_j \circ \psi_{j-i}$ .

Let  $f(t, \cdot)$  for  $t \in I$  be defined as in (b). For  $t > 1$  we set  $f(t, \cdot) = f(1, \cdot)$  and for  $t < 0$  we set  $f(t, \cdot) = f(0, \cdot)$ . Then we define  $g(t, \cdot) = f(tN_0/N, \cdot)$  for all  $t \in \mathbf{R}$  and we set  $U = \bigcap \{U_j : j = 1, 2, \dots, N\}$ . Clearly (a) and (b) are satisfied.

*Proof of (c):* First observe that  $\varphi_j \in \mathcal{D}(K_j + \mathbf{U}(0, \varepsilon/\Delta))$  for each  $j = 1, 2, \dots, N$  due to 7.11(a). Since  $K_j + \mathbf{B}(0, \varepsilon/\Delta) \subseteq G_\varepsilon$  we see that  $\psi_j \in \mathcal{D}(G_\varepsilon)$ .

Fix  $t \in I$  and choose  $j \in \mathcal{P}$  such that  $j \leq tN \leq j+1$ . We have

$$f(t, \cdot) = (s(tN - j)\varphi_{j+1} + (1 - s(tN - j))\text{id}_{\mathbf{R}^n}) \circ \psi_j.$$

Since  $\varphi_{j+1} \in \mathcal{D}(K_j + \mathbf{U}(0, \varepsilon/\Delta))$  and  $K_{j+1} + \mathbf{U}(0, \varepsilon/\Delta)$  is convex we see that

$$s(tN - j)\varphi_{j+1} + (1 - s(tN - j))\text{id}_{\mathbf{R}^n} \in \mathcal{D}(K_{j+1} + \mathbf{U}(0, \varepsilon/\Delta));$$

hence,  $f(t, \cdot) \in \mathcal{D}(G_\varepsilon)$ .

*Proof of (d):* Since  $\mathcal{F}$  is admissible for  $j \in \{1, 2, \dots, N\}$  and  $L \in \mathbf{CX}(\mathcal{F})$  exactly one of the following options holds

- $L \cap K_j = \emptyset$  and then  $\varphi_j[L] = L$  by 7.11(b);
- $L \cap K_j \neq \emptyset$  and  $\mathbf{l}(L) \geq \frac{1}{2}\mathbf{l}(K_j)$  and  $L \cap \text{Int}_c(K_j) = \emptyset$  and  $\varphi_j[L] \subseteq L$  by 7.11(f);
- $L \cap K_j \neq \emptyset$  and  $L \cap \text{Int}_c(K_j) \neq \emptyset$  and  $\dim(L) > \dim(K_j)$  and  $\mathbf{l}(L) \geq \mathbf{l}(K_j)$ , because  $K_j \in \mathbf{CX}(\mathcal{F})$ , and  $\varphi_j[L] \subseteq L$  by 7.7 and 7.11(f).

In consequence, if  $L \in \mathbf{CX}(\mathcal{F})$ , then  $\psi_j[L] \subseteq L$  for  $j \in \{1, \dots, N\}$  and, since  $L$  is convex, we obtain  $f(t, \cdot)[L] \subseteq L$  for  $t \in I$ . In particular

$$(41) \quad \varphi_j[L] \subseteq L \quad \text{for } j \in \{1, 2, \dots, N\} \text{ and } L \in \mathbf{CX}(\mathcal{F}).$$

*Proof of (e):* For  $j \in \{1, \dots, N\}$  set

$$\alpha(j) = \min\{i : \dim(K_i) = \dim(K_j)\} \quad \text{and} \quad \beta(j) = \max\{i : \dim(K_i) = \dim(K_j)\}.$$

For  $j = 1, \dots, N$  set

$$\mathcal{C}_j = \{L \in \mathbf{K}_* : \exists i \in \mathcal{P} \quad \alpha(j) \leq i \leq j \text{ and } L \text{ is a face of } K_i\}.$$

We shall prove by induction the following claim:

$$(42) \quad \left| \begin{array}{l} \text{for each } j = 1, 2, \dots, N_0 \text{ if } k = \dim(K_j), \text{ then} \\ \psi_j[U] \cap G_0 \subseteq \bigcup (\mathcal{C}_j \cap \mathbf{K}_{k-1}) \cup \bigcup ((\mathcal{C}_{\beta(j)} \sim \mathcal{C}_j) \cap \mathbf{K}_k). \end{array} \right.$$

If  $j = 1$ , then  $k = n$  and  $\psi_1 = \varphi_1$  and claim (42) follows from 7.11(c)(f).

Assume  $j \in \{2, 3, \dots, N_0\}$  and  $k = \dim(K_j)$ . By inductive hypothesis, if  $j > \alpha(j)$ , then

$$\begin{aligned}\psi_{j-1}[U] \cap F &= \bigcup \left( (\mathcal{C}_{j-1} \cap \mathbf{K}_{k-1}) \cup ((\mathcal{C}_{\beta(j)} \sim \mathcal{C}_{j-1}) \cap \mathbf{K}_k) \right) \cap F \cap \psi_{j-1}[U] \\ &\subseteq \bigcup (\mathcal{C}_{j-1} \cap \mathbf{K}_{k-1}) \cup (\psi_{j-1}[U] \cap K_j) \cup \left( \bigcup ((\mathcal{C}_{\beta(j)} \sim \mathcal{C}_j) \cap \mathbf{K}_k) \cap F \right)\end{aligned}$$

and if  $j = \alpha(j)$ , then  $j - 1 = \beta(j - 1)$  and

$$\begin{aligned}\psi_{j-1}[U] \cap F &= \bigcup (\mathcal{C}_{\beta(j-1)} \cap \mathbf{K}_k) \cap F \cap \psi_{j-1}[U] = \bigcup (\mathcal{C}_{\beta(j)} \cap \mathbf{K}_k) \cap F \cap \psi_{j-1}[U] \\ &\subseteq (\psi_{j-1}[U] \cap K_j) \cup \left( \bigcup ((\mathcal{C}_{\beta(j)} \sim \mathcal{C}_j) \cap \mathbf{K}_k) \cap F \right).\end{aligned}$$

Claim (42) follows now by the following observations:

- if  $j > \alpha(j)$  and  $Q \in \mathcal{C}_{j-1} \cap \mathbf{K}_{k-1}$ , then  $Q \cap \text{Int}_c(K_j) = \emptyset$  so  $\varphi_j[Q] \subseteq Q$  by (41);
- if  $\alpha(k) \leq j \leq \beta(j)$ , we have  $\varphi_j[\psi_{j-1}[U] \cap K_j] \subseteq \text{Bdry}_c(K_j) \subseteq \bigcup (\mathcal{C}_j \cap \mathbf{K}_{k-1})$  by 7.11(c);
- if  $\alpha(k) \leq j \leq \beta(j)$  and  $Q \in (\mathcal{C}_{\beta(j)} \sim \mathcal{C}_j) \cap \mathbf{K}_k$ , then  $\varphi_j[Q] \subseteq Q$  by (41).

*Proof of (f):* This follows by (e) and the construction.

*Proof of (g):* To prove (35) take  $x \in G_\varepsilon \sim G_0$  and note that are at most  $\Delta$  maps amongst  $\varphi_1, \dots, \varphi_N$  which move the point  $x$  so, recalling 7.11(g), we get

$$(43) \quad \|Df(t, \cdot)(x)\| \leq \|D\psi_{j+1}(x)\| + \|D\psi_j(x)\| \leq 2\Gamma_{7.11}^\Delta \quad \text{whenever } j \leq Nt \leq j+1.$$

To prove (36) choose a cube  $Q \in \mathcal{F}$  and  $i \in \{1, 2, \dots, l\}$ . Set

$$\tilde{Q}_\varepsilon = \tilde{Q} + \mathbf{U}(0, \varepsilon) = \bigcup \{R \in \mathcal{F} : R \cap Q \neq \emptyset\} + \mathbf{U}(0, \varepsilon).$$

Observe that for  $j \in \{1, 2, \dots, N_0\}$  we have

$$\eta_{Q,j}^{-1}[\tilde{Q}_\varepsilon] = \tilde{Q}_\varepsilon \quad \text{and} \quad \zeta_{Q,j}^{-1}[\tilde{Q}_\varepsilon] = \tilde{Q}_\varepsilon.$$

Therefore, we may write

$$\begin{aligned}(44) \quad \int_{\tilde{Q}_\varepsilon \cap \Sigma_i} \|D\eta_{Q,j}\|^{m_i} d\mathcal{H}^{m_i} &\leq \int_{\eta_{Q,j}^{-1}[\tilde{Q}_\varepsilon]} \|(D\zeta_{Q,j}) \circ \eta_{Q,j-1}\|^{m_i} \|D\eta_{Q,j-1}\|^{m_i} d\mathcal{H}^{m_i} \llcorner \Sigma_i \\ &= \int_{\zeta_{Q,j}^{-1}[\tilde{Q}_\varepsilon]} \|D\zeta_{Q,j}\|^{m_i} d\nu_{Q,j} = \int_{\tilde{Q}_\varepsilon} \|D\zeta_{Q,j}\|^{m_i} d\nu_{Q,j}.\end{aligned}$$

If  $\zeta_{Q,j} = \text{id}_{\mathbf{R}^n}$ , then we obtain

$$(45) \quad \int_{\tilde{Q}_\varepsilon} \|D\zeta_{Q,j}\|^{m_i} d\nu_{Q,j} = \nu_{Q,j}(\tilde{Q}_\varepsilon) = \int_{\tilde{Q}_\varepsilon \cap \Sigma_i} \|D\eta_{Q,j-1}\|^{m_i} d\mathcal{H}^{m_i}.$$

If  $\zeta_{Q,j} = \varphi_k$  for some  $k \in \{1, 2, \dots, N_0\}$ , then  $K_k \subseteq \tilde{Q}_\varepsilon$  and if  $K_k$  is a face of some cube  $R \in \mathbf{K}_*$  with  $\dim R > \dim K_k$ , then  $\text{spt } \nu_{Q,j} \cap \text{Int}_c(R) = \emptyset$ ; thus, employing 7.11(b)(g)(h), we get

$$\begin{aligned}(46) \quad \int_{\tilde{Q}_\varepsilon} \|D\zeta_{Q,j}\|^{m_i} d\nu_{Q,j} &= \int_{\tilde{Q}_\varepsilon \cap K_k} \|D\varphi_k\|^{m_i} d\nu_{Q,j} + \int_{\tilde{Q}_\varepsilon \sim K_k} \|D\varphi_k\|^{m_i} d\nu_{Q,j} \\ &\leq \Gamma_{7.11} \nu_{Q,j}(\tilde{Q}_\varepsilon) = \Gamma_{7.11} \int_{\tilde{Q}_\varepsilon \cap \Sigma_i} \|D\eta_{Q,j-1}\|^{m_i} d\mathcal{H}^{m_i}.\end{aligned}$$

Now, the second case (when  $\zeta_{Q,j} = \varphi_k$  for some  $k \in \{1, 2, \dots, N_0\}$ ) can happen at most  $\Delta$  times so combining (46), (45), and (44)

$$\int_{\tilde{Q}_\varepsilon \cap \Sigma_i} \|D\eta_{Q,j}\|^{m_i} d\mathcal{H}^{m_i} \leq \Gamma_{7.11}^\Delta \mathcal{H}^{m_i}(\Sigma_i \cap \tilde{Q}_\varepsilon).$$

We may now finish the proof of (36) by writing

$$\begin{aligned} \int_{\Sigma_i \cap Q} \|Dg(t, \cdot)\|^{m_i} d\mathcal{H}^{m_i} &\leq 2^{m_i-1} \int_{\Sigma_i \cap Q} \|D\psi_{j+1}(x)\|^{m_i} + \|D\psi_j(x)\|^{m_i} d\mathcal{H}^{m_i} \\ &= 2^{m_i-1} \int_{\Sigma_i \cap Q} \|D\eta_{Q,j+1}(x)\|^{m_i} + \|D\eta_{Q,j}(x)\|^{m_i} d\mathcal{H}^{m_i} \leq 2^{m_i} \Gamma_{7.11}^\Delta \mathcal{H}^{m_i}(\Sigma_i \cap \tilde{Q}_\varepsilon), \end{aligned}$$

whenever  $j \in \{1, 2, \dots, N_0\}$  and  $t \in \mathbf{R}$  are such that  $j \leq N_0 t \leq j+1$ .

To prove (37) we write

$$\begin{aligned} \int_{\Sigma_i \cap G_\varepsilon} \|Dg(t, \cdot)\|^{m_i} d\mathcal{H}^{m_i} &\leq \sum_{Q \in \mathcal{A}} \int_{\Sigma_i \cap Q} \|Dg(t, \cdot)\|^{m_i} d\mathcal{H}^{m_i} + \int_{\Sigma_i \cap G_\varepsilon \sim G_0} \|Dg(t, \cdot)\|^{m_i} d\mathcal{H}^{m_i} \\ &\leq 2^{m_i} \Gamma_{7.11}^\Delta \sum_{Q \in \mathcal{A}} \mathcal{H}^{m_i}(\Sigma_i \cap \tilde{Q}_\varepsilon) + 2^{m_i} \Gamma_{7.11}^{\Delta m_i} \mathcal{H}^{m_i}(\Sigma_i \cap G_\varepsilon \sim G_0) \\ &\leq 2^{m_i} \Delta^2 \Gamma_{7.11}^{\Delta m_i} \mathcal{H}^{m_i}(\Sigma_i \cap G_\varepsilon). \end{aligned}$$

To prove (38) and (39) note that for  $Q \in \mathcal{F}$ , by construction (in particular by (f)),

$$\mathcal{H}^{m_i}(f(1, \cdot)[\Sigma_i \cap Q] \cap G_0) \leq \mathcal{H}^{m_i}(g(1, \cdot)[\Sigma_i \cap Q] \cap G_0)$$

and, recalling (d) and (43),

$$\mathcal{H}^{m_i}(f(1, \cdot)[\Sigma_i \cap Q] \sim G_0) \leq 2^{m_i} \Gamma_{7.11}^{\Delta m_i} \mathcal{H}^{m_i}(\Sigma_i \cap Q \sim G_0).$$

Assume now that  $\Sigma_i$  is  $(\mathcal{H}^{m_i}, m_i)$  rectifiable. Then  $I \times \Sigma_i$  is  $(\mathcal{H}^{m_i+1}, m_i+1)$  rectifiable by [Fed69, 3.2.23] and we may use the area formula [Fed69, 3.2.20] to prove (40). Define

$$A_j = G_\varepsilon \cap \bigcup \{R \in \mathcal{F} : R \cap K_{j+1} \neq \emptyset\}.$$

Let  $\tau = (1, 0) \in \mathbf{R} \times \mathbf{R}^n$  be the “time direction” and set  $g_t(x) = g(t, x)$  for  $(t, x) \in \mathbf{R} \times \mathbf{R}^n$ . Observe that for  $(t, x) \in I \times \Sigma$  and  $j \in \mathcal{P}$  such that  $j \leq tN_0 \leq j+1$  we have

$$\begin{aligned} (\mathcal{H}^{m_i+1} \llcorner (I \times \Sigma_i), m_i+1) \text{ ap } J_{m_i+1} g(t, x) &\leq |Dg(t, x)\tau| \cdot \|Dg_t(x)\|^{m_i} \\ &\leq 2N_0 |\varphi_{j+1}(\psi_j(x)) - \psi_j(x)| \cdot \|Dg_t(x)\|^{m_i}; \end{aligned}$$

hence, using [Fed69, 3.2.20, 3.2.23], (b), (d), 7.11(b)(e)

$$\begin{aligned} (47) \quad \mathcal{H}^{m_i+1}(g[I \times (\Sigma_i \cap G_\varepsilon)]) &\leq \int_0^1 \int_{\Sigma_i \cap G_\varepsilon} |Dg(t, x)\tau| \cdot \|Dg_t(x)\|^{m_i} d\mathcal{H}^{m_i}(x) d\mathcal{L}^1(t) \\ &\leq \sum_{j=0}^{N_0-1} \int_{j/N_0}^{(j+1)/N_0} \int_{\Sigma_i \cap G_\varepsilon} 2N_0 |\varphi_{j+1}(\psi_j(x)) - \psi_j(x)| \cdot \|Dg_t(x)\|^{m_i} d\mathcal{H}^{m_i}(x) d\mathcal{L}^1(t) \\ &\leq 2^{m_i} N_0 \sqrt{n} \delta \sum_{j=0}^{N_0-1} \int_{j/N_0}^{(j+1)/N_0} d\mathcal{L}^1(t) \int_{\Sigma_i \cap A_j} \|D\psi_{j+1}(x)\|^{m_i} + \|D\psi_j(x)\|^{m_i} d\mathcal{H}^{m_i}(x) \\ &\leq 2^{m_i} \sqrt{n} \delta 2\Delta^2 \Gamma_{7.11}^{\Delta m_i} \mathcal{H}^{m_i}(\Sigma_i \cap G_\varepsilon). \end{aligned}$$

If  $m = m_1 > m_l$ , then  $N = N_0$  and  $f = g$  and there is nothing more to prove. Otherwise, we have  $m = m_1 = \dots = m_l$  and  $g[I \times (\Sigma_i \cap G_\varepsilon)] = f[I' \times (\Sigma_i \cap G_\varepsilon)]$ , where  $I' = [0, N_0/N]$ . Hence, we need to estimate  $\mathcal{H}^{m+1}(f[(I \sim I') \times (\Sigma_i \cap G_\varepsilon)])$ . Observe, that

$$f[(I \sim I') \times (\Sigma_i \cap G_0)] \subseteq \bigcup \mathbf{CX}(\mathcal{F}) \cap \mathbf{K}_m,$$

so  $\mathcal{H}^{m+1}(f[(I \sim I') \times (\Sigma_i \cap G_0)]) = 0$ . On the other hand  $\|Df_t(x)\| \leq \Gamma_{7.11}$  for  $x \in G_\varepsilon \sim G_0$  and  $t \in I \sim I'$  so  $\mathcal{H}^{m+1}(f[(I \sim I') \times (\Sigma_i \cap G_\varepsilon \sim G_0)])$  can be estimated as in (47).  $\square$

We finish this section with a small lemma that allows to apply 4.3 to the mapping constructed in 7.13.

**7.14 Lemma.** *If  $f \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$  and  $U \subseteq \mathbf{R}^n$  and  $\dim_{\mathcal{H}}(f[U]) \leq m$ , then  $\dim \operatorname{im} Df(x) \leq m$  for  $x \in U$ .*

*Proof.* Assume there exists a point  $x \in U$  such that  $\dim \operatorname{im} Df(x) = k > m$ . Define  $L = \operatorname{im} Df(x) \in \mathbf{G}(n, k)$  and set  $g = L_{\sharp} \circ f$ . Observe that

$$\mathcal{H}^k(g[\mathbf{B}(x, r)]) \leq \mathcal{H}^k(f[\mathbf{B}(x, r)]) \quad \text{for } r > 0.$$

Moreover, since  $f(y) = f(x) + Df(x)(y - x) + o(|x - y|)$  we see that for small enough  $r > 0$  we have

$$f(x) + Df(x)[\mathbf{B}(0, r/2)] \subseteq g[\mathbf{B}(x, r)].$$

Hence, for some  $r > 0$  we obtain  $\mathcal{H}^k(f[\mathbf{B}(x, r)]) > 0$  which contradicts  $\dim_{\mathcal{H}}(f[U]) \leq m$ .  $\square$

## 8 Slicing varifolds by continuously differentiable functions

We recall the theory developed by Almgren in [Alm76, I.3] and [Alm65, §7].

In this sections we shall always assume  $U \subseteq \mathbf{R}^n$  is open,  $m, n, \nu \in \mathbf{Z}$  are such that  $0 \leq \nu \leq m < n$ ,  $V \in \mathbf{V}_m(U)$ ,  $f \in \mathcal{C}^1(U, \mathbf{R}^\nu)$  is proper, and  $\pi : \mathbf{R}^n \times \mathbf{G}(n, m) \rightarrow \mathbf{R}^n$  is the projection onto the first factor.

**8.1 Definition.** Whenever  $\beta \in \mathcal{K}(U \times \mathbf{G}(n, m - \nu))$  and  $\varphi \in \mathcal{K}(\mathbf{R}^\nu)$  we set

$$(V, f)(\beta) = \int_{\{(x, S) \in U \times \mathbf{G}(n, m) : \|\bigwedge_\nu Df(x) \circ S_{\sharp}\| > 0\}} \beta(x, S \cap \ker Df(x)) \|\bigwedge_\nu Df(x) \circ S_{\sharp}\| dV(x, S),$$

$$\mu_\beta(\varphi) = (V, f)(\beta \cdot (\varphi \circ f \circ \pi)).$$

It was shown in [Alm76, I.3(2)] that  $(V, f) \in \mathbf{V}_{m-\nu}(U)$  and  $\mu_\beta$  is a Radon measure for each  $\beta \in \mathcal{K}(U \times \mathbf{G}(n, m - \nu))$ .

**8.2 Definition.** The *slice of  $V$  with respect to  $f$  at  $t \in \mathbf{R}^\nu$*  is the varifold  $\langle V, f, t \rangle \in \mathbf{V}_{m-\nu}(U)$ , satisfying

$$\langle V, f, t \rangle(\beta) = \lim_{r \downarrow 0} \frac{\mu_\beta(\mathbf{B}(t, r))}{\mathcal{L}^\nu(\mathbf{B}(t, r))}, \quad \text{whenever } \beta \in \mathcal{K}(U \times \mathbf{G}(n, m - \nu)).$$

**8.3 Remark.** By [Alm76, I.3(2)], there exists  $\langle V, f, t \rangle \in \mathbf{V}_{m-\nu}(U)$  and, since  $f$  is proper,  $\text{spt } \|\langle V, f, t \rangle\|$  is compact for  $\mathcal{L}^\nu$  almost all  $t \in \mathbf{R}^\nu$ . Next, we view  $\{V \in \mathbf{V}_{m-\nu}(U) : \text{spt } \|V\| \text{ is compact}\}$  as a subset of the vectorspace (cf. [Fed69, 2.5.19])

$$\{W \in \mathcal{K}(U \times \mathbf{G}(n, m - \nu))^* : \text{spt } W \text{ is compact}\}, \quad \text{with the norm} \\ \||W\|| = \sup\{W(\beta) : \beta \in \mathcal{K}(U \times \mathbf{G}(n, m - \nu)), \sup \text{im } |\beta| \leq 1, \text{Lip } \beta \leq 1\}.$$

Then  $\{W \in \mathcal{K}(U \times \mathbf{G}(n, m - \nu))^* : \text{spt } W \text{ is compact}\}$  becomes a separable normed vectorspace such that the norm topology coincides with the weak topology.

**8.4 Definition** (cf. [Alm76, I.3(3)]). The *Lebesgue set* of the slicing operator  $\langle V, f, \cdot \rangle$  is the set of those  $t \in \mathbf{R}^\nu$  for which

$$\lim_{r \downarrow 0} r^{-\nu} \int_{\mathbf{B}(t, r)} \||\langle V, f, s \rangle - \langle V, f, t \rangle\|| \, d\mathcal{L}^\nu(s) = 0.$$

**8.5 Remark.** Note that  $\mathcal{L}^\nu$  almost all  $t \in \mathbf{R}^\nu$  are Lebesgue points of  $\langle V, f, \cdot \rangle$ ; see [Alm76, I.3(3)] for the proof.

**8.6 Remark.** Recalling [Alm76, I.3(4)] we see that if  $S \subseteq U$  is  $(\mathcal{H}^m, m)$  rectifiable and  $\mathcal{H}^m$  measurable and bounded, then

$$\langle \mathbf{v}(S), f, t \rangle = \mathbf{v}(S \cap f^{-1}\{t\}) \in \mathbf{RV}_{m-\nu}(U) \quad \text{for } \mathcal{L}^\nu \text{ almost all } t \in \mathbf{R}^\nu.$$

Next, we define the product of a varifold with a cube as in [Alm76, I.3(5)]. Products of more general varifolds were described also in [KM15, §3].

**8.7 Definition.** If  $l \in \mathbf{Z}$ , and  $l \geq 1$ , and  $I = \{t \in \mathbf{R} : 0 \leq t \leq 1\}$ , and  $j_l : \mathbf{R}^l \rightarrow \mathbf{R}^l \times \mathbf{R}^n$  and  $j_n : \mathbf{R}^n \rightarrow \mathbf{R}^l \times \mathbf{R}^n$  are injections, then

$$(\mathbf{v}(I^l) \times V)(\alpha) = \int_{I^l} \int \alpha((t, x), j_l[\mathbf{R}^l] + j_n[T]) \, dV(x, T) \, d\mathcal{L}^l(t),$$

for  $\alpha \in \mathcal{K}(\mathbf{R}^l \times U \times \mathbf{G}(l + n, l + m))$ .

**8.8 Definition.** For  $t \in \mathbf{R}$  and  $\delta \in (0, 1)$  and  $\rho : \mathbf{R}^n \rightarrow \mathbf{R}$  we define the functions

$$i_t : \mathbf{R}^n \rightarrow \mathbf{R} \times \mathbf{R}^n, \quad s_\delta : \mathbf{R} \rightarrow \mathbf{R}, \quad K_{\rho, t, \delta} : U \rightarrow \mathbf{R} \times U,$$

by requiring that  $s_\delta$  is of class  $\mathcal{C}^\infty$  and

$$s_\delta(\tau) = \tau \quad \text{for } \delta \leq \tau \leq 1 - \delta, \quad s_\delta(\tau) = 0 \quad \text{for } \tau \leq 0, \quad s_\delta(\tau) = 1 \quad \text{for } \tau \geq 1, \\ 0 \leq s'_\delta(\tau) \leq 1 + \delta \quad \text{for } \tau \in \mathbf{R}, \quad i_t(x) = (t, x) \quad \text{for } x \in \mathbf{R}^n, \\ K_{\rho, t, \delta}(x) = (s_\delta((t - \rho(x))/\delta), x) \quad \text{for } x \in \mathbf{R}^n.$$

**8.9 Lemma.** Let  $V \in \mathbf{V}_m(U)$  and  $\rho \in \mathcal{C}^1(U, \mathbf{R})$  be a proper map, and  $\iota \in (0, \infty)$ , and  $t \in \mathbf{R}$  be a Lebesgue point of  $\langle V, \rho, \cdot \rangle$  such that  $V \llcorner \{(x, S) \in \mathbf{R}^n \times \mathbf{G}(n, m) : t - \iota \leq \rho(x) \leq t\} \in \mathbf{RV}_m(U)$  and

$$(48) \quad \lim_{\delta \downarrow 0} \|V\|(\{x \in U : t - \delta \leq \rho(x) < t\}) = 0.$$

Set  $V_0 = V \llcorner \{(x, S) \in U \times \mathbf{G}(n, m) : \rho(x) \geq t\}$  and  $V_1 = V \llcorner \{(x, S) \in U \times \mathbf{G}(n, m) : \rho(x) < t\}$ . Then

$$\lim_{\delta \downarrow 0} K_{\rho, t, \delta} \# V = i_0 \# V_0 + i_1 \# V_1 + \mathbf{v}(I) \times \langle V, \rho, t \rangle \in \mathbf{V}_m(\mathbf{R} \times U).$$

*Proof.* Since  $\rho$  and  $t$  are fixed we abbreviate  $K_\delta = K_{\rho,t,\delta}$ . For  $\delta \in (0, 1)$  define

$$\begin{aligned} V_{1,\delta} &= V \sqcup \{(x, S) \in U \times \mathbf{G}(n, m) : \rho(x) < t - \delta\}, \\ V_{2,\delta} &= V \sqcup \{(x, S) \in U \times \mathbf{G}(n, m) : t - \delta \leq \rho(x) < t\}. \end{aligned}$$

Clearly

$$K_\delta \# V = i_0 \# V_0 + i_1 \# V_{1,\delta} + K_\delta \# V_{2,\delta}$$

and  $\lim_{\delta \downarrow 0} i_1 \# V_{1,\delta} = i_1 \# V_1$  so it suffices to prove that  $\lim_{\delta \downarrow 0} K_\delta \# V_{2,\delta} = \mathbf{v}(I) \times \langle V, \rho, t \rangle$ . To this end it is enough to show that  $\lim_{\delta \downarrow 0} \|K_\delta \# V_{2,\delta} - \mathbf{v}(I) \times \langle V, \rho, t \rangle\| = 0$ .

Let  $j_i : \mathbf{R} \rightarrow \mathbf{R} \times U$  and  $j_n : U \rightarrow \mathbf{R} \times U$  be injections and let  $\pi : U \times \mathbf{G}(n, m-1) \rightarrow U$  be the projection onto the first factor. For  $\|V\|$  almost all  $x$  we define  $T$ ,  $R_\delta$ ,  $P$ ,  $J_{K,\delta}$ , and  $J_\rho$  by requiring

$$\begin{aligned} T(x) &= \text{Tan}^m(\|V\|, x) \in \mathbf{G}(n, m), \quad R_\delta(x) = DK_\delta(x)[T(x)] \in \mathbf{G}(n+1, m), \\ P(x) &= j_1[\mathbf{R}] + j_n[T(x) \cap \ker D\rho(x)] \in \mathbf{G}(n+1, m), \\ J_{K,\delta}(x) &= (\|V\|, m) \text{ap } J_m K_\delta(x) \in \mathbf{R}, \quad J_\rho(x) = (\|V\|, m) \text{ap } J_1 \rho(x) \in \mathbf{R}. \end{aligned}$$

Whenever  $x \in \text{dmn } T$  and  $Q \in \mathbf{G}(n, m-1)$  and  $\tau \in [0, 1]$  we also set

$$\begin{aligned} \gamma_\tau(x, Q) &= ((\tau, x), j_1[\mathbf{R}] + j_n[Q]) \in (\mathbf{R} \times U) \times \mathbf{G}(n+1, m), \\ \psi_{\tau,\delta}(x) &= ((s_\delta(\tau), x), R_\delta(x)) \in (\mathbf{R} \times U) \times \mathbf{G}(n+1, m), \\ W_\delta &= K_\delta \# V_{2,\delta} - \mathbf{v}(I) \times \langle V, \rho, t \rangle \in \mathcal{K}(\mathbf{R} \times U \times \mathbf{G}(n+1, m))^*. \end{aligned}$$

Let  $\varepsilon \in (0, 1/2)$ . If  $\|\langle V, \rho, t \rangle\| > 0$ , then assume additionally that  $\varepsilon \leq 2^{-5} \|\langle V, \rho, t \rangle\|^{-1}$ . Find  $\delta_0 \in (0, 1)$  such that for all  $\delta \in (0, \delta_0)$

$$(49) \quad \delta < \min\{2^{-5}\varepsilon(\|\langle V, \rho, t \rangle\| + 2^{-4}\varepsilon)^{-1}, \varepsilon^2/2, \iota\},$$

$$(50) \quad \frac{1}{\delta} \int_{t-\delta}^t \|\langle V, \rho, \tau \rangle - \langle V, \rho, t \rangle\| \, d\mathcal{L}^1(\tau) \leq 2^{-4}\varepsilon,$$

$$(51) \quad \|V\|(\{x \in U : t - \delta \leq \rho(x) < t\}) \leq (1 + \varepsilon^{-4})^{-1} 2^{-4}\varepsilon.$$

Such  $\delta_0 > 0$  exists because  $t$  is a Lebesgue point of  $\langle V, \rho, \cdot \rangle$  and we assumed (48). It follows from (50), applied to  $[t - \delta^2, t]$  and  $[t - \delta + \delta^2, t]$  and  $[t - \delta, t]$ , and from (49) that

$$(52) \quad \int_{\{\tau \in I : \tau < \delta \text{ or } \tau > 1 - \delta\}} \|\langle V, \rho, t - \delta\tau \rangle\|(U) \, d\mathcal{L}^1(\tau) \leq 2(\|\langle V, \rho, t \rangle\| + 2^{-4}\varepsilon)\delta \leq 2^{-4}\varepsilon.$$

For any  $\alpha \in \mathcal{K}(\mathbf{R} \times U \times \mathbf{G}(n+1, m))$  such that  $\sup \text{im } |\alpha| \leq 1$  and  $\text{Lip } \alpha \leq 1$ , employing the co-area formula [Fed69, 3.2.22], we get

$$\begin{aligned} (53) \quad |W_\delta(\alpha)| &= \left| \int_0^1 \langle V, \rho, t - \delta\tau \rangle (\alpha \circ \psi_{\delta,\tau} \circ \pi \cdot (\delta J_{K,\delta}/J_\rho) \circ \pi) - \langle V, \rho, t \rangle (\alpha \circ \gamma_\tau) \, d\mathcal{L}^1(\tau) \right| \\ &\leq \left| \int_0^1 \langle V, \rho, t - \delta\tau \rangle (\alpha \circ \psi_{\delta,\tau} \circ \pi \cdot (\delta J_{K,\delta}/J_\rho) \circ \pi - \alpha \circ \gamma_\tau) \, d\mathcal{L}^1(\tau) \right| \\ &\quad + \left| \int_0^1 (\langle V, \rho, t - \delta\tau \rangle - \langle V, \rho, t \rangle) (\alpha \circ \gamma_\tau) \, d\mathcal{L}^1(\tau) \right| = B_1(\alpha, \delta) + B_2(\alpha, \delta). \end{aligned}$$

Since  $\text{Lip } \gamma_\tau = 1$  for  $\tau \in [0, 1]$ , we have by (50)

$$(54) \quad B_2(\alpha, \delta) \leq \int_0^1 \|\langle V, \rho, t - \delta\tau \rangle - \langle V, \rho, t \rangle\| \, d\mathcal{L}^1(\tau) \leq 2^{-4}\varepsilon.$$

To estimate  $B_1(\alpha, \delta)$  we set

$$\begin{aligned} X &= \{(x, S) \in \text{dmn } T \times \mathbf{G}(n, m-1) : S = T(x) \cap \ker D\rho(x) \in \mathbf{G}(n, m-1)\}, \\ A_1(\delta) &= \{(x, S) \in X : J_\rho(x) \leq \varepsilon^{-2}\delta \text{ and } \delta J_{K,\delta}(x) \geq J_\rho(x)\}, \\ A_2(\delta) &= \{(x, S) \in X : J_\rho(x) > \varepsilon^{-2}\delta \text{ and } \delta J_{K,\delta}(x) \geq J_\rho(x)\}, \\ A_3(\delta) &= \{(x, S) \in X : \delta J_{K,\delta}(x) < J_\rho(x)\}. \end{aligned}$$

Clearly  $\sum_{i=1}^3 \langle V, \rho, s \rangle \lrcorner A_i(\delta) = \langle V, \rho, s \rangle$  for  $s \in [t - \delta, t]$ . We estimate first the third part. Straightforward computations (cf. [Alm76, I.3(1)]) show that if  $(x, S) \in X$  and  $\tau \in [0, 1]$  and  $\delta \in (0, 1)$  and  $\rho(x) = t - \delta\tau$ , then, setting  $v = T(x)_\natural(\text{grad } \rho(x)) \in S^\perp \cap T$ ,

$$(55) \quad \begin{aligned} R_\delta(x) &= \text{span}\{j_1(1)s'_\delta(\tau)|v|/\delta + j_n(v/|v|)\} + j_n[T(x) \cap \ker D\rho(x)], \quad J_\rho(x) = |v|, \\ \delta J_{K,\delta}(x) &= (\delta^2 + s'_\delta(\tau)^2 J_\rho(x)^2)^{1/2}, \quad \|R_\delta(x)_\natural - P(x)_\natural\| = J_{K,\delta}(x)^{-1}, \end{aligned}$$

$$(56) \quad \gamma_\tau(x, S) = ((\tau, x), P(x)).$$

Hence, recalling 8.8 we see that  $(t - \rho(x))/\delta \in [0, \delta] \cup (1 - \delta, 1]$  whenever  $(x, S) \in A_3(\delta)$  so

$$(57) \quad \begin{aligned} &\left| \int_0^1 (\langle V, \rho, t - \delta\tau \rangle \lrcorner A_3(\delta)) (\alpha \circ \psi_{\delta,\tau} \circ \pi \cdot (\delta J_{K,\delta}/J_\rho) \circ \pi - \alpha \circ \gamma_\tau) \, d\mathcal{L}^1(\tau) \right| \\ &\leq 2 \int_{\{\tau \in I : \tau < \delta \text{ or } \tau > 1 - \delta\}} \|\langle V, \rho, t - \delta\tau \rangle\|(U) \, d\mathcal{L}^1(\tau) \leq 2^{-3}\varepsilon \quad \text{by (52)}. \end{aligned}$$

For  $(x, S) \in A_1(\delta)$  we have  $J_{K,\delta}(x) \leq (1 + \varepsilon^{-4}\delta^2)^{1/2} \leq 1 + \varepsilon^{-4}$  and  $\delta J_{K,\delta}(x)/J_\rho(x) \geq 1$  so, using the co-area formula [Fed69, 3.2.22] and  $\sup \text{im } |\alpha| \leq 1$ , we obtain

$$(58) \quad \begin{aligned} &\left| \int_0^1 (\langle V, \rho, t - \delta\tau \rangle \lrcorner A_1(\delta)) (\alpha \circ \psi_{\delta,\tau} \circ \pi \cdot (\delta J_{K,\delta}/J_\rho) \circ \pi - \alpha \circ \gamma_\tau) \, d\mathcal{L}^1(\tau) \right| \\ &\leq 2 \sup \text{im } |\alpha| (1 + \varepsilon^{-4}) \|V\|(\{x \in U : t - \delta \leq \rho(x) < t\}) \leq 2^{-3}\varepsilon \quad \text{by (51)}. \end{aligned}$$

To deal with  $A_2(\delta)$  first observe that  $\delta J_{K,\delta}(x)/J_\rho(x) \geq 1$  and  $J_\rho(x) > \delta\varepsilon^{-2}$  and  $\varepsilon \leq 1/2$  imply

$$s'_\delta(\tau)^2 \geq 1 - \delta^2/J_\rho(x)^2 \geq 1 - \varepsilon^4 \geq 1/2,$$

where  $\tau = (t - \rho(x))/\delta$ . Therefore, by (56) and (55) and (49),

$$\begin{aligned} M &= \sup\{|\psi_{\delta,\tau}(x) - \gamma_\tau(x, S)| : (x, S) \in A_2(\delta), \delta\tau = t - \rho(x)\} \\ &= \delta + \sup\{\|R_\delta(x)_\natural - P(x)_\natural\| : (x, S) \in A_2(\delta), \delta\tau = t - \rho(x)\} \\ &\leq \varepsilon^2/2 + \delta/(\delta^2 + 1/2\delta^2/\varepsilon^4)^{1/2} \leq 2\varepsilon^2. \end{aligned}$$

If  $(x, S) \in A_2(\delta)$ , then  $\delta \leq \varepsilon^2 J_\rho(x)$  so  $\delta J_{K,\delta}(x)/J_\rho(x) \leq 1 + \varepsilon^4$  and, using  $\text{Lip } \alpha \leq 1$ ,

$$(59) \quad \begin{aligned} &\left| \int_0^1 (\langle V, \rho, t - \delta\tau \rangle \lrcorner A_2(\delta)) (\alpha \circ \psi_{\delta,\tau} \circ \pi \cdot (\delta J_{K,\delta}/J_\rho) \circ \pi - \alpha \circ \gamma_\tau) \, d\mathcal{L}^1(\tau) \right| \\ &\leq (\varepsilon^4 + M) \int_0^1 \|\langle V, \rho, t - \delta\tau \rangle\|(U) \, d\mathcal{L}^1(\tau) \leq 4\varepsilon^2 \int_0^1 \|\langle V, \rho, t - \delta\tau \rangle\|(U) \, d\mathcal{L}^1(\tau). \end{aligned}$$

Recall that  $\varepsilon < 1/2$  and if  $\|\langle V, \rho, t \rangle\| > 0$ , we assumed  $\varepsilon \leq 2^{-5} \|\langle V, \rho, t \rangle\|^{-1}$ . Thus, using (50)

$$(60) \quad 4\varepsilon^2 \int_0^1 \|\langle V, \rho, t - \delta\tau \rangle\|(U) d\mathcal{L}^1(\tau) \leq \varepsilon(4\varepsilon \|\langle V, \rho, t \rangle\| + 2^{-2}\varepsilon^2) \leq 2^{-2}\varepsilon.$$

Finally, combining (53), (54), (57), (58), (59), (60) we see that  $|W_\delta(\alpha)| \leq \varepsilon$  for  $\delta \in (0, \delta_0)$ . Since  $\delta_0$  was chosen independently of  $\alpha$  we have  $\|W_\delta\| \leq \varepsilon$  for  $\delta \in (0, \delta_0)$ .  $\square$

**8.10 Corollary.** *Assume  $a, b \in \mathbf{R}$  are such that  $V \llcorner \{(x, S) \in \mathbf{R}^n \times \mathbf{G}(n, m) : a \leq \rho(x) \leq b\} \in \mathbf{RV}_m(U)$ . Then for  $\mathcal{L}^1$  almost all  $t \in (a, b)$*

$$\lim_{\delta \downarrow 0} K_{\rho, t, \delta} \# V = i_0 \# V_0 + i_1 \# V_1 + \mathbf{v}(I) \times \langle V, \rho, t \rangle \in \mathbf{V}_m(\mathbf{R} \times U).$$

## 9 Density ratio bounds

The main result 9.3 of this section gives lower and upper bounds on the density ratios of  $\|V\|$  for any  $V$  which minimises a bounded  $\mathcal{C}^0$  integrand  $F$  (not necessarily elliptic). Our proof follows the ideas presented in [Alm68, 2.9(b2)(b3), 3.2(a)(b), 3.4(2) last paragraphs on pp. 347 and 348] as well as in [Fle66, 7.8, 8.2].

Let  $a \in U \subseteq \mathbf{R}^n$  be fixed,  $\rho(x) = |x - a|$ , and  $M(r) = \|V\| \{x \in \mathbf{R}^n : \rho(x) < r\}$  for  $r > 0$ . The key point is to prove the differential inequalities (77) and (79). Assume  $M'(r_0)$  exists and is finite for some  $r_0 > 0$  – this holds for  $\mathcal{L}^1$  almost all  $r_0 \in (0, \infty)$ . Recall that  $V$  is a limit of some sequence of the form  $\{\mathbf{v}(S_i \cap U) : i \in \mathcal{P}\}$ , where  $S_i$  belong to a good class. Our strategy is to first choose a compact set  $S \subseteq \mathbf{R}^n$  such that  $\mathbf{v}(S \cap U)$  is weakly close to  $V$ , then construct an admissible deformation  $D$  (using the deformation theorem 7.13) of  $S$  such that the  $\mathcal{H}^m(D[S])$  can be estimated in terms of  $M'(r_0)$ , and finally use minimality of  $V$  to derive estimates on  $M(r_0)$ .

More precisely we proceed as follows. We choose a compact set  $S \subseteq \mathbf{R}^n$  so that the  $4d$ -neighbourhood of  $S \cap \rho^{-1}\{r_0\}$  has  $\mathcal{H}^m$  measure controlled roughly by  $dM'(r_0)$ , where  $d > 0$  is a small number. This is possible because the mass  $V \mapsto \|V\|(\mathbf{R}^n)$  is continuous on the space of varifolds supported in a fixed compact set. Then, we use the deformation theorem 7.13 to “project” the part of  $S$  lying inside  $2d$ -neighbourhood of  $\rho^{-1}\{r_0\}$  onto some  $m$  dimensional cubical complex and we denote the deformed set  $R$ . After this step the part of  $R$  lying in a  $2d$ -neighbourhood of  $\rho^{-1}\{r_0\}$  is  $(\mathcal{H}^m, m)$  rectifiable (as a finite sum of  $m$  dimensional cubes) and, moreover, its measure is still controlled by  $dM'(r_0)$  due to the first part of 7.13(g) which holds for non-rectifiable sets. Next, we use the co-area formula (valid on rectifiable sets) together with the Chebyshev inequality and 8.5 to find some  $r_1 > 0$  in the  $d$ -neighbourhood of  $r_0$  so that the  $\mathcal{H}^{m-1}$  measure of the slice  $R \cap \rho^{-1}\{r_1\}$  is controlled by  $M'(r_0)$  and  $r_1$  is a Lebesgue point of the slicing operator  $\langle \mathbf{v}(R), \rho, \cdot \rangle$  and  $R \cap \rho^{-1}\{r_1\}$  is  $(\mathcal{H}^{m-1}, m-1)$  rectifiable.

To prove 9.2(a) we cover the slice  $R \cap \rho^{-1}\{r_1\}$  with cubes of equal size  $\varepsilon > 0$  and apply the deformation theorem 7.13 again to obtain a map  $g_2 : I \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ . We choose  $\varepsilon$  so big that the whole slice  $R \cap \rho^{-1}\{r_1\}$  does not fill, after the deformation, a single  $m$ -dimensional cube, which amounts to setting  $\varepsilon \approx M'(r_0)^{1/(m-1)}$ . This ensures that  $g_2(1, \cdot)[R \cap \rho^{-1}\{r_1\}]$  lies in some  $m-2$  dimensional cubical complex. Since  $r_1$  is a Lebesgue point of  $\langle \mathbf{v}(R), \rho, \cdot \rangle$  we may perform a blow-up of the slice using 8.9. Then we make use of the smoothness



of  $g_2$  to argue that the push-forward  $g_{2\#}$  is continuous on the space of varifolds so that we can compose  $g_2$  with the blow-up map  $K_\delta$  and pass to the limit. Next, we estimate the  $\mathcal{H}^m$  measure of the blow-up limit only by *one* term, namely the  $\mathcal{H}^m$  measure of the image of the whole deformation of the slice, i.e.,  $\mathcal{H}^m(g_2[I \times R \cap \rho^{-1}\{r_1\}])$ . The other terms drop out because  $g_2$  deformed our slice into an  $m - 2$  dimensional set. Since  $R \cap \rho^{-1}\{r_1\}$  is  $(\mathcal{H}^{m-1}, m - 1)$  rectifiable we can use the second part of 7.13(g) to estimate  $\mathcal{H}^m(g_2[I \times R \cap \rho^{-1}\{r_1\}])$  by  $\varepsilon M'(r_0) \approx M'(r_0)^{m/(m-1)}$ . Finally, we make use of continuity of the mass to choose one deformation from the blow-up sequence (without passing to the limit) for which the desired estimate still holds.

To prove 9.2(a) we proceed similarly but this time we choose  $\varepsilon \approx \iota > 0$  arbitrarily, we deform the part of  $R$  lying in the ball  $\{x \in \mathbf{R}^n : \rho(x) < r_1\}$  rather than just the slice  $R \cap \rho^{-1}\{r_1\}$ , and we get two terms in the final estimate. The first term corresponds to  $\iota M'(r_0)$  analogously as before and the second can be estimated brutally by the  $\mathcal{H}^m$  measure of the sum of all  $m$  dimensional cubes from  $\mathbf{K}_m$  touching the ball  $\{x \in \mathbf{R}^n : \rho(x) \leq r_0 + d + 6\iota\sqrt{n}\}$ . Later, we use scaling to show that this second term depends only on  $\iota$  and, in 9.3, we choose a specific  $\iota$  depending only on  $n, m$ , and  $F$ .

We begin with a technical lemma. The estimates in 9.1(a)(b) contain an additional term which includes the parameter  $d$  and which shall be later absorbed by other terms. The parameter  $b$  below shall be set to  $M'(r_0)$  in most cases.

**9.1 Lemma.** *Let  $S \subseteq \mathbf{R}^n$ , and  $a \in \mathbf{R}^n$ , and  $b, d \in (0, \infty)$ , and  $r_0 \in (0, \infty)$ . Set  $\rho(x) = |x - a|$ . Assume  $\mathcal{H}^m(\text{Clos } S \cap \mathbf{B}(a, r_0 + 4d)) < \infty$  and*

$$\mathcal{H}^m(\{x \in S : r_0 - 4d \leq \rho(x) < r_0 + 4d\}) < 9bd,$$

(a) *There exists a deformation  $D \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R}^n)$  such that*

$$(61) \quad D \in \mathfrak{D}(\{x \in \mathbf{R}^n : \rho(x) < r_0 + 4d + 40\sqrt{n}(\Gamma_{7.13}^2 b)^{1/(m-1)}\}),$$

$$\mathcal{H}^m(D[\{x \in S : \rho(x) < r_0 + 4d\}]) \leq 9\Gamma_{7.13}bd + 50\Gamma_{7.13}^{2m/(m-1)}b^{m/(m-1)}.$$

(b) *Suppose  $\iota \in (0, \infty)$  and  $N \in \mathbf{Z}$  satisfy  $2^{-N-1} < \iota \leq 2^{-N}$ . There exists a deformation  $F \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R}^n)$  such that setting*

$$(62) \quad \Delta(\rho, \iota, r_0, d) = 2^{-Nm} \mathcal{H}^0(\{K \in \mathbf{K}_m(N) : \{x \in K : \rho(x) < r_0 + d + 6\iota\sqrt{n}\} \neq \emptyset\})$$

*there holds*

$$(63) \quad F \in \mathfrak{D}(\{x \in \mathbf{R}^n : \rho(x) < r_0 + 4d + 8\iota\sqrt{n}\})$$

$$\mathcal{H}^m(F[\{x \in S : \rho(x) < r_0 + 4d\}]) \leq 9\Gamma_{7.13}bd + 10\Gamma_{7.13}^{2m/(m-1)}\iota b + \Delta(\rho, \iota, r_0, d).$$

*Proof.* For brevity define  $A(d) = \{x \in \mathbf{R}^n : r_0 - d \leq \rho(x) < r_0 + d\}$  for  $d \in (0, \infty)$ . Set  $\varepsilon_1 = (5n)^{-1/2}d$  and find  $N_1 \in \mathbf{Z}$  such that  $2^{-N_1-1} < \varepsilon_1 \leq 2^{-N_1}$ . Define

$$\mathcal{A}_1 = \{Q \in \mathbf{K}_n(N_1) : Q \cap A(2d) \neq \emptyset\}.$$

Note that  $\mathcal{A}_1$  is finite. Apply 7.13 with  $2, m, m, S, S \cap A(2d), 2^{-N_1-4}, \mathbf{K}_n(N_1), \mathcal{A}_1$  in place of  $l, m_1, m_2, \Sigma_1, \Sigma_2, \varepsilon, \mathcal{F}, \mathcal{A}$  to obtain the map  $g_1 : I \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  called "f" there. Observe that

$$\bigcup \mathcal{A}_1 + \mathbf{B}(0, 2^{-N_1-4}) \subseteq A(4d) \quad \text{and} \quad S \cap A(2d) \subseteq A(2d) \subseteq \text{Int}(\bigcup \mathcal{A}_1).$$

In particular, it follows from 7.13(a) that

$$(64) \quad g_1(t, x) = x \quad \text{whenever } \rho(x) \geq r_0 + 4d \text{ and } t \in I.$$

Define  $R = g_1(1, \cdot)[S]$  and note that  $R \cap \text{Int}(\bigcup \mathcal{A}_1)$  is a finite sum of  $m$  dimensional cubes; in particular it is  $(\mathcal{H}^m, m)$  rectifiable. Observe also that if  $x \in R \cap A(d)$ , then there exists an  $n$  dimensional cube  $K \in \mathcal{A}_1$  such that  $x \in K$  and there exists  $y \in S$  such that  $g_1(y) = x$  and  $y \in K$  due to 7.13(d). Hence,  $|g(y) - y| \leq 2\varepsilon_1 \sqrt{n} < d$  and, since  $\text{Lip } \rho \leq 1$ , we get  $y \in A(2d)$ . Therefore,

$$(65) \quad R \cap A(d) \subseteq g[S \cap A(2d)] \quad \text{and} \quad \mathcal{H}^m(R \cap A(d)) < 9\Gamma_{7.13}bd.$$

by 7.13(g) and (9.1). Since  $R \cap A(d) \subseteq \text{Int}(\bigcup \mathcal{A}_1)$  is  $(\mathcal{H}^m, m)$  rectifiable we may employ the co-area formula [Fed69, 3.2.22] together with  $\text{Lip } \rho \leq 1$  to obtain

$$9\Gamma_{7.13}bd > \mathcal{H}^m(R \cap A(d)) \geq \int_{r_0-d}^{r_0+d} \mathcal{H}^{m-1}(R \cap \rho^{-1}\{t\}) d\mathcal{L}^1(t).$$

Thus, the Chebyshev inequality gives

$$\mathcal{L}^1(\{t \in (r_0 - d, r_0 + d) : \mathcal{H}^{m-1}(R \cap \rho^{-1}\{t\}) \geq 5\Gamma_{7.13}b\}) \leq \frac{9}{10}2d.$$

Now we see that there exists  $r_1 \in (r_0 - d, r_0 + d)$  such that

$$(66) \quad \mathcal{H}^{m-1}(R \cap \rho^{-1}\{r_1\}) < 5\Gamma_{7.13}b$$

and  $r_1$  is a Lebesgue point of the slicing operator  $\langle \mathbf{v}(R), \rho, \cdot \rangle$  (see 8.4) and  $\langle \mathbf{v}(R), \rho, r_1 \rangle \in \mathbf{RV}_{m-1}(\mathbf{R}^n)$  (see 8.6). For  $\delta \in (0, r_1 - r_0 + d)$  let  $K_\delta = K_{\rho, r_1, \delta} : \mathbf{R}^n \rightarrow I \times \mathbf{R}^n$  be defined as in 8.8. Since  $R \cap A(d) \subseteq \text{Int}(\bigcup \mathcal{A}_1)$  is a finite sum of  $m$  dimensional cubes we can apply 8.9 to see that

$$(67) \quad \lim_{\delta \downarrow 0} K_\delta \# \mathbf{v}(R) = i_0 \# \mathbf{v}(\{x \in R : \rho(x) \geq r_1\}) + i_1 \# \mathbf{v}(\{x \in R : \rho(x) < r_1\}) \\ + \mathbf{v}([0, 1]) \times \langle \mathbf{v}(R), \rho, r_1 \rangle \in \mathbf{V}_m(\mathbf{R} \times \mathbf{R}^n),$$

where  $i_0$  and  $i_1$  are defined as in 8.8.

*Proof of (a):* For brevity, if  $K \in \mathbf{K}$ , let us define  $\hat{K}$  to be the  $n$  dimensional cube with the same centre as  $K$  and side length three times as long as  $K$ . Choose  $\varepsilon_2 \in (0, \infty)$  and  $N_2 \in \mathbf{Z}$  so that

$$(68) \quad \varepsilon_2^{m-1} = \Gamma_{7.13} \mathcal{H}^{m-1}(R \cap \rho^{-1}\{r_1\}) < 5\Gamma_{7.13}^2 b \quad \text{and} \quad 2^{-N_2-1} < \varepsilon_2 \leq 2^{-N_2}.$$

Define

$$B = R \cap \rho^{-1}\{r_1\} \quad \text{and} \quad \mathcal{A}_2 = \{K \in \mathbf{K}_n(N_2) : \hat{K} \cap \rho^{-1}\{r_1\} \neq \emptyset\}.$$

Apply 7.13 with 1,  $m-1$ ,  $B$ ,  $2^{-N_2-4}$ ,  $\mathbf{K}_n(N_2)$ ,  $\mathcal{A}_2$  in place of  $l$ ,  $m_1$ ,  $\Sigma_1$ ,  $\varepsilon$ ,  $\mathcal{F}$ ,  $\mathcal{A}$  to obtain the map  $g_2 : I \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  called "f" there. We define  $h : I \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  by setting

$$h(t, x) = g_2(2t, x) \quad \text{for } t \in [0, 1/2] \text{ and } x \in \mathbf{R}^n, \\ h(t, x) = s_{1/100}(2-2t)g_2(1, x) \quad \text{for } t \in (1/2, 1] \text{ and } x \in \mathbf{R}^n,$$

where  $s_{1/100}$  is the function defined in 8.8. Due to our choice of  $\varepsilon_2$  we know, from 7.13(g), that  $\mathcal{H}^{m-1}(g_2[B]) < \mathcal{H}^{m-1}(K)$  for any  $(m-1)$  dimensional cube  $K \in \mathbf{K}_{m-1}(N_2)$ . We see also that  $g_2[B] \subseteq \text{Int}(\bigcup \mathcal{A}_2)$  because  $g_2[B] \subseteq \bigcup \{K \in \mathbf{K}_n(N_2) : K \cap B \neq \emptyset\}$  by 7.13(d). Hence,

$$g_2[R \cap \rho^{-1}\{r_1\}] = g_2[B \cap \bigcup \mathcal{A}_2] \subseteq \bigcup \mathbf{K}_{m-2}(N_2),$$

by 7.13(f), and we obtain

$$(69) \quad h(0, \cdot)_{\#} \mathbf{v}(\{x \in R : \rho(x) \geq r_1\}) = \mathbf{v}(\{x \in R : \rho(x) \geq r_1\}),$$

$$(70) \quad h(1, \cdot)_{\#} \mathbf{v}(\{x \in R : \rho(x) < r_1\}) = 0,$$

$$(71) \quad h_{\#} \mathbf{v}(I \times R \cap \rho^{-1}\{r_1\}) = g_2_{\#} \mathbf{v}(I \times R \cap \rho^{-1}\{r_1\}).$$

Since  $h$  is of class  $\mathcal{C}^\infty$  the push-forward  $h_{\#}$  is continuous on  $\mathbf{V}_m(\mathbf{R} \times \mathbf{R}^n)$  so using (67) together with (69), (70), (71)

$$\begin{aligned} \lim_{\delta \downarrow 0} (h \circ K_\delta)_{\#} \mathbf{v}(\{x \in R : \rho(x) < r_0 + 4d\}) &= \mathbf{v}(\{x \in R : r_1 \leq \rho(x) < r_0 + 4d\}) \\ &\quad + g_2_{\#} \mathbf{v}(I \times R \cap \rho^{-1}\{r_1\}). \end{aligned}$$

Since  $\rho$  is proper we can find a continuous function  $\gamma : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$  with compact support such that  $\{(t, x) \in I \times \mathbf{R}^n : \rho(x) \leq r_0 + 4d\} \subseteq \text{Int } \gamma^{-1}\{1\}$  and use it as a test function for the weak convergence. Thus, recalling that  $\langle \mathbf{v}(R), \rho, r_1 \rangle \in \mathbf{RV}_{m-1}(\mathbf{R}^n)$  and employing 7.13(g), (66), (65) we obtain

$$\begin{aligned} (72) \quad \lim_{\delta \downarrow 0} \mathcal{H}^m((h \circ K_\delta)[\{x \in R : \rho(x) < r_0 + 4d\}]) \\ \leq \mathcal{H}^m(\{x \in R : r_1 \leq \rho(x) < r_0 + 4d\}) + 2\varepsilon_2 \Gamma_{7.13} \mathcal{H}^{m-1}(R \cap \rho^{-1}\{r_1\}) \\ < 9\Gamma_{7.13} b d + 10\varepsilon_2 \Gamma_{7.13}^2 b. \end{aligned}$$

For  $\delta \in (0, r_1 - r_0 + d)$  define  $D_\delta = h \circ K_\delta \circ g_1(1, \cdot)$ . Using (64), (68), (72) we can find  $\delta_0 \in (0, r_1 - r_0 + d)$  such that for all  $\delta \in (0, \delta_0]$

$$\mathcal{H}^m(D_\delta[\{x \in S : \rho(x) < r_0 + 4d\}]) \leq 9\Gamma_{7.13} b d + 50\Gamma_{7.13}^{2m/(m-1)} b^{m/(m-1)}.$$

Observe that

$$\bigcup \mathcal{A}_2 + \mathbf{B}(0, 2^{-N_2-4}) \subseteq A(d + 8\sqrt{n}(5\Gamma_{7.13}^2 b)^{1/(m-1)});$$

hence,  $D_\delta \in \mathfrak{D}(\text{conv } A(4d + 40\sqrt{n}(\Gamma_{7.13}^2 b)^{1/(m-1)}))$  for  $\delta \in (0, \delta_0)$  so setting  $D = D_{\delta_0}$  finishes the proof of (a).

*Proof of (b):* Recall that if  $K \in \mathbf{K}$ , then  $\hat{K}$  denotes the  $n$  dimensional cube with the same centre as  $K$  and side length three times as long as  $K$ . Let  $\iota \in (0, \infty)$  and  $N \in \mathbf{Z}$  satisfy  $2^{-N-1} < \iota \leq 2^{-N}$ . Set

$$C = \{x \in R : \rho(x) < r_1\} \quad \text{and} \quad \mathcal{A}_3 = \{K \in \mathbf{K}_n(N) : \hat{K} \cap \{x \in \mathbf{R}^n : \rho(x) < r_1\} \neq \emptyset\}.$$

Apply 7.13 with  $2, m, m-1, C, B, 2^{-N-4}, \mathbf{K}_n(N), \mathcal{A}_3$  in place of  $l, m_1, m_2, \Sigma_1, \Sigma_2, \varepsilon, \mathcal{F}, \mathcal{A}$  to obtain the map  $g_3 : I \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  called “ $g$ ” there. Since  $C \subseteq \bigcup \{K \in \mathbf{K}_n(N) : K \cap \{x \in \mathbf{R}^n : \rho(x) < r_1\} \neq \emptyset\}$  we see that  $g_3(1, \cdot)[C] \subseteq \text{Int}(\bigcup \mathcal{A}_3)$ , by 7.13(d), and conclude from 7.13(e) that

$$(73) \quad \mathcal{H}^m(g_3(1, \cdot)[C]) \leq \mathcal{H}^m(\bigcup \mathbf{K}_m(N) \cap \bigcup \mathcal{A}_3) \leq \Delta(\rho, \iota, r_0, d).$$

Therefore, using (67) and 7.13(g), (65), (66), (73) we get

$$\begin{aligned} \lim_{\delta \downarrow 0} \mathcal{H}^m((g_3 \circ K_\delta)[\{x \in R : \rho(x) < r_0 + 4d\}]) \\ \leq \mathcal{H}^m(\{x \in R : r_1 \leq \rho(x) < r_0 + 4d\}) + \mathcal{H}^m(g_3[I \times B]) + \mathcal{H}^m(g_3(1, \cdot)[C]) \\ \leq 9\Gamma_{7.13}bd + 10\iota\Gamma_{7.13}^2b + \Delta(\rho, \iota, r_0, d). \end{aligned}$$

Hence, there exists  $\delta_0 \in (0, r_1 - r_0 + d)$  such that  $F = g_3 \circ K_{\delta_0} \circ g_1(1, \cdot)$  satisfies the estimate claimed in (b). Moreover, we see that

$$\bigcup \mathcal{A}_3 + \mathbf{B}(0, 2^{-N-4}) \subseteq \{x \in \mathbf{R}^n : \rho(x) < r_0 + d + 8\iota\sqrt{n}\};$$

hence,  $F \in \mathfrak{D}(\{x \in \mathbf{R}^n : \rho(x) < r_0 + 4d + 8\iota\sqrt{n}\})$ .  $\square$

Now, we can prove the pivotal differential inequalities (77) and (79). There is one technical difficulty that needs to be taken care of. To be able to employ minimality of  $V$  we need our deformations to be admissible in an open set  $U \subseteq \mathbf{R}^n$ . In particular, in the proof of (77) we need to perform a deformation onto cubes of side length roughly  $M'(r_0)^{1/(m-1)}$  which might be arbitrarily big. This turns out not to be a problem since big values of the derivative  $M'$  cannot spoil the lower bound on  $M$ . More precisely, we will later use the upper bound on  $M$  proven in 9.3(a) to overcome this difficulty. For the time being we just assume in 9.2(a) that some upper bound on  $M$  holds.

**9.2 Lemma.** *Assume*

$$\begin{aligned} (74) \quad & U \subseteq \mathbf{R}^n \text{ is open, } \quad a \in U, \quad \mathcal{C} \text{ is a good class in } U, \quad \{S_i : i \in \mathscr{P}\} \subseteq \mathcal{C}, \\ & \rho(x) = |x - a|, \quad r_0, \iota \in (0, \infty), \quad V = \lim_{i \rightarrow \infty} \mathbf{v}(S_i \cap U) \in \mathbf{V}_m(U), \\ & F \text{ is a } \mathcal{C}^0 \text{ integrand, } \quad \Phi_F(V) = \lim_{i \rightarrow \infty} \Phi_F(S_i \cap U) = \inf\{\Phi_F(T \cap U) : T \in \mathcal{C}\}, \\ & M(t) = \|V\| \mathbf{U}(a, t) \quad \text{for } t \in \mathbf{R}, \quad M'(r_0) \text{ exists and is finite,} \\ & \alpha = \inf F[\mathbf{B}(a, \text{dist}(a, \mathbf{R}^n \sim U))] > 0, \quad \beta = \sup F[\mathbf{B}(a, \text{dist}(a, \mathbf{R}^n \sim U))] < \infty, \\ (75) \quad & \Gamma = \Gamma(n, m, F, U, a) = 240\Gamma_{7.13}^{2m/(m-1)}\beta/\alpha. \end{aligned}$$

(a) *If  $M(r_0) \leq \gamma r_0^m$  or  $M'(r_0) \leq (\gamma/\Gamma)^{1-1/m} r_0^{m-1}$  for some  $\gamma \in (0, \infty)$ , and*

$$(76) \quad \kappa = \kappa(n, m, \Gamma, \gamma) = 1 + (\gamma/\Gamma)^{1/m} (4\Gamma_{7.13}^{(m+1)/(m-1)} + 40\sqrt{n}\Gamma_{7.13}^{2/(m-1)}),$$

*and  $\mathbf{B}(a, \kappa r_0) \subseteq U$ , then*

$$(77) \quad M(r_0) \leq \Gamma M'(r_0)^{m/(m-1)}.$$

(b) *There exists  $\gamma = \gamma(n, m, \iota) \in (1, \infty)$  such that setting*

$$(78) \quad \kappa = \kappa(n, m, \iota) = 1 + \iota (4\Gamma_{7.13}^{(m+1)/(m-1)} + 8\sqrt{n})$$

*if  $\mathbf{B}(a, \kappa r_0) \subseteq U$ , then*

$$(79) \quad \frac{M(r_0)}{r_0^m} \leq \gamma + \Gamma \iota \frac{M'(r_0)}{r_0^{m-1}}.$$

*Proof.* Define  $A(d) = \{x \in \mathbf{R}^n : r_0 - d \leq \rho(x) < r_0 + d\}$  for  $d \in (0, \infty)$ . Choose  $b, d \in (0, \infty)$  so that

$$(80) \quad M'(r_0) \leq b, \quad [r_0 - 4d, r_0 + 4d] \subseteq (0, \infty), \\ \|V\|(A(4d)) = M(r_0 + 4d) - M(r_0 - 4d) < 9bd, \quad \|V\|(\text{Bdry } A(4d)) = 0.$$

$$(81) \quad d < \Gamma_{7.13}^{(m+1)/(m-1)} \min\{b^{1/(m-1)}, \iota r_0\}.$$

It follows from (80) that  $\lim_{i \rightarrow \infty} \|v(S_i)\|(A(4d)) = \|V\|(A(4d))$ ; hence, recalling (74), we see that there exists  $S \in \{S_i : i \in \mathcal{P}\}$  such that

$$(82) \quad 0 \leq \Phi_F(S \cap U) - \Phi_F(V) < \frac{1}{4}\alpha M(r_0), \quad \mathcal{H}^m(S \cap A(4d)) < 9bd,$$

$$(83) \quad M(r_0) < 2\|v(S)\|(\{x \in \mathbf{R}^n : \rho(x) < r_0\}).$$

*Proof of (a):* If  $M(r_0) \leq \gamma r_0^m$  and  $M'(r_0) > (\gamma/\Gamma)^{1-1/m} r_0^{m-1}$ , then (77) follows trivially. Thus, we may and shall assume that  $M'(r_0) < (\gamma/\Gamma)^{1-1/m} r_0^{m-1}$ . Suppose also  $b \leq (\gamma/\Gamma)^{1-1/m} r_0^{m-1}$  and define  $\kappa$  by (76).

Now, recalling (61) and (81), we apply 9.1(a) together with (81) to obtain the deformation  $D \in \mathfrak{D}(\mathbf{U}(a, \kappa r_0))$  such that

$$\mathcal{H}^m(D[\{x \in S : \rho(x) < r_0 + 4d\}]) \leq 59\Gamma_{7.13}^{2m/(m-1)} b^{m/(m-1)}.$$

Since  $D \in \mathfrak{D}(U)$  we have  $\Phi_F(V) \leq \Phi_F(D[S] \cap U)$ . Using (83) and (82), and noting that  $D(x) = x$  whenever  $\rho(x) \geq r_0 + 4d$  we see that

$$(84) \quad \frac{1}{4}\alpha M(r_0) \leq \alpha \mathcal{H}^m(\{x \in S : \rho(x) < r_0 + 4d\}) - \frac{1}{4}\alpha M(r_0) \\ \leq \Phi_F(\{x \in S : \rho(x) < r_0 + 4d\}) + (\Phi_F(V) - \Phi_F(S \cap U)) \\ = \Phi_F(V) - \Phi_F(D[\{x \in S \cap U : \rho(x) \geq r_0 + 4d\}]) \\ \leq \Phi_F(D[\{x \in S : \rho(x) < r_0 + 4d\}]) \leq 59\beta\Gamma_{7.13}^{2m/(m-1)} b^{m/(m-1)}.$$

Recalling (75), the definition of  $\Gamma$ , we see that

$$M(r_0) \leq \Gamma b^{m/(m-1)}.$$

Clearly  $M$  is non-decreasing so  $M'(r_0) \geq 0$ . If  $M'(r_0) > 0$ , then the proof of (a) is finished by setting  $b = M'(r_0)$ . If  $M'(r_0) = 0$ , then we may choose  $b > 0$  arbitrarily small to obtain  $M(r_0) = 0 \leq \Gamma M'(r_0) = 0$ .

*Proof of (b):* From (a) we already know that if  $M'(r_0) = 0$ , then  $M(r) = 0$  so we may assume  $M'(r_0) > 0$  and set  $b = M'(r_0)$ . Define

$$\bar{S} = \mu_{1/r_0} \circ \tau_{-a}[S], \quad \bar{M}(s) = \|(\mu_{1/r_0} \circ \tau_{-a})_{\#} V\|(\mathbf{U}(0, s)) = r_0^{-m} M(sr_0) \text{ for } s \in (0, \infty), \\ \bar{\rho}(x) = \rho \circ \tau_a(x) = |x| \text{ for } x \in \mathbf{R}^n, \quad \bar{b} = \bar{M}'(1) = \frac{M'(r_0)}{r_0^{m-1}}, \quad \bar{d} = \frac{d}{r_0} < \Gamma_{7.13}^{(m+1)/(m-1)} \iota.$$

Apply 9.1(b) with  $\bar{S}, \bar{d}, \bar{\rho}, \bar{b}, \iota$  in place of  $S, d, \rho, b, \iota$  to obtain the deformation  $F \in \mathfrak{D}(\mathbf{U}(0, \kappa))$ , where  $\kappa = \kappa(n, m, \iota)$  is defined by (78). Combining (81) and (63) we see that

$$\mathcal{H}^m(F[\bar{S} \cap \mathbf{U}(0, 1 + 4\bar{d})]) \leq 19\Gamma_{7.13}^{2m/(m-1)} \iota \bar{M}'(1) + \Delta(\iota),$$

where  $\Delta(\iota) = \Delta(\bar{\rho}, \iota, 1, \bar{d})$  is defined by (62). Set  $L = \tau_a \circ \mu_{r_0} \circ F \circ \mu_{1/r_0} \circ \tau_{-a}$ . Since  $B(a, \kappa r_0) \subseteq U$  we have  $L \in \mathfrak{D}(U)$  so  $\Phi_F(V) \leq \Phi_F(L[S])$  and we can compute as in (84)

$$\begin{aligned} \frac{\alpha}{4\beta} M(r_0) &\leq \mathcal{H}^m(L[S \cap \mathbf{U}(a, r_0(1 + 4\bar{d})])) \\ &= r_0^m \mathcal{H}^m(F[\bar{S} \cap \mathbf{U}(0, 1 + 4\bar{d})]) \leq r_0^m (19\Gamma_{7.13}^{2m/(m-1)} \iota \bar{M}'(1) + \Delta(\iota)) \\ &= 19\Gamma_{7.13}^{2m/(m-1)} \iota r_0 M'(r_0) + r_0^m \Delta(\iota). \end{aligned}$$

Hence, we may set  $\gamma = 4\beta/\alpha\Delta(\iota)$ . □

### 9.3 Theorem. Assume

$$\begin{aligned} U &\subseteq \mathbf{R}^n \text{ is open, } \mathcal{C} \text{ is a good class in } U, \quad \{S_i : i \in \mathcal{P}\} \subseteq \mathcal{C}, \\ V &= \lim_{i \rightarrow \infty} \mathbf{v}(S_i \cap U) \in \mathbf{V}_m(U), \quad F \text{ is a bounded } \mathcal{C}^0 \text{ integrand,} \\ \Phi_F(V) &= \lim_{i \rightarrow \infty} \Phi_F(S_i \cap U) = \inf\{\Phi_F(T \cap U) : T \in \mathcal{C}\} < \infty, \\ \Delta &= \sup\{\Gamma_{9.2}(n, m, F, U, x) : x \in \text{spt } \|V\|\}, \quad \iota = (2m\Delta)^{-1}, \\ \kappa &= \kappa(n, m, F, V, U) = \kappa_{9.2(b)}(n, m, \iota), \quad a \in \text{spt } \|V\| \subseteq U, \quad r_0 = \text{dist}(a, \mathbf{R}^n \sim U)/\kappa. \end{aligned}$$

Then  $\Delta, \iota \in (0, \infty)$  and the following statements hold.

(a) There exists  $\Gamma = \Gamma(n, m, F, V, U) \in (0, \infty)$  such that for all  $r \in (0, r_0)$

$$r^{-m} \|V\| \mathbf{U}(a, r) \leq \max\{\Gamma, r_0^{-m} \|V\| \mathbf{U}(a, r_0)\}.$$

(b) Define  $\lambda = \lambda(n, m, F, V, U, a) = \max\{\kappa, \kappa_{9.2(a)}(n, m, \Gamma_{9.2}(n, m, F, U, a), \gamma)\}$ , where

$$\gamma = \gamma(n, m, F, V, U, a) = \max\{\Gamma_{9.3(a)}(n, m, F, V, U), r_0^{-m} \|V\| \mathbf{U}(a, r_0)\}.$$

For all  $r \in (0, \text{dist}(a, \mathbf{R}^n \sim U)/\lambda)$  we have

$$r^{-m} \|V\| \mathbf{U}(a, r) \geq m^{-m} \Gamma_{9.2}(n, m, F, U, a)^{1-m}.$$

*Proof.* Since  $F$  is bounded, it attains its supremum and infimum. Thus, recalling the definition of  $\Gamma_{9.2}(n, m, F, U, \cdot)$  we see that  $0 < \Delta < \infty$ .

*Proof of (a):* Let  $a \in \text{spt } \|V\|$  and  $r \in (0, r_0)$ , where  $r_0 = \text{dist}(a, \mathbf{R}^n \sim U)/\kappa$ . Set  $M(s) = \|V\| \mathbf{U}(a, s)$  for  $s \in (0, \infty)$ . Define  $\gamma = \gamma_{9.2(b)}(n, m, \iota)$  and  $\Gamma = 2\gamma/\alpha(m)$ . For each  $s \in (0, r_0)$  for which  $M'(s)$  exists and is finite we may apply 9.2(b) to see that

$$(85) \quad s^{-m} M(s) \leq \gamma + \Delta \iota s^{-(m-1)} M'(s).$$

Now we proceed as in [Fle66, 8.2]. Choose  $\eta > \Gamma$  and assume there exists  $r_1 \in (0, r_0)$  satisfying  $M(r_1) > \eta \alpha(m) r_1^m$ . Let  $r_2 \in [r_1, r_0]$  be the largest number in  $[r_1, r_0]$  such that  $M(s) \geq \eta \alpha(m) s^m$  for  $s \in [r_1, r_2]$ . Since  $M$  is non-decreasing we see immediately that  $r_2 > r_1$ . Using (85) and the definitions of  $\iota$  and  $\Gamma$ , we obtain for  $\mathcal{L}^1$  almost all  $s \in [r_1, r_2]$

$$M(s) \leq s^m \gamma + \Delta \iota s M'(s) < \frac{1}{2} M(s) + \frac{1}{2m} s M'(s).$$

Hence,  $mM(s) < sM'(s)$  for  $\mathcal{L}^1$  almost all  $s \in [r_1, r_2]$  which implies that

$$(s^{-m} M(s))' > 0 \quad \text{for } \mathcal{L}^1 \text{ almost all } s \in [r_1, r_2].$$

Using [Fed69, 2.9.19] for each  $s_1, s_2 \in [r_1, r_2]$  with  $s_1 < s_2$  we obtain

$$0 < \int_{s_1}^{s_2} (t^{-m} M(t))' d\mathcal{L}^1(t) \leq s_2^{-m} M(s_2) - s_1^{-m} M(s_1),$$

which shows that  $s^{-m} M(s)$  is increasing for  $s \in [r_1, r_2]$ ; thus,  $r_2 = r_0$ . Since  $\eta > \Gamma$  could be arbitrary the claim is proven.

*Proof of (b):* For each  $s \in (0, \text{dist}(a, \mathbf{R}^n \sim U)/\lambda)$  for which  $M'(s)$  exists and is finite we may apply 9.2(a) to see that

$$(M^{1/m})'(s) \geq m^{-1} \Gamma_{9.2}^{(1-m)/m}.$$

Employing [Fed69, 2.9.19] we find out that  $M'(s)$  exists and is finite for  $\mathcal{L}^1$  almost all  $s \in (0, r_0)$  and that

$$\|V\| \mathbf{U}(a, r)^{1/m} = M(r)^{1/m} \geq \int_0^r (M^{1/m})'(s) d\mathcal{L}^1(s) \geq m^{-1} \Gamma_{9.2}^{(1-m)/m} r. \quad \square$$

**9.4 Corollary.** *Let  $F, V$ , and  $U$  be as in 9.3 and  $\delta > 0$ . There exist  $\Gamma = \Gamma(n, m, F, V, U, \delta) > 1$  and  $\kappa = \kappa(n, m, F, V, U, \delta) > 1$  such that for all  $x \in \text{spt } \|V\| \subseteq U$  and  $r \in (0, \infty)$  satisfying  $r < \text{dist}(x, \mathbf{R}^n \sim U)/\kappa$  and  $\text{dist}(x, \mathbf{R}^n \sim U) > \delta$  there holds*

$$\Gamma^{-1} r^m \leq \|V\| \mathbf{B}(x, r) \leq \Gamma r^m.$$

*In particular, for all  $x \in \text{spt } \|V\| \cap E$  we have*

$$0 < \Theta_*^m(\|V\|, x) \leq \Theta^{*m}(\|V\|, x) < \infty.$$

*Using [Fed69, 2.10.19(1)(3), 2.1.3(5)] and Borel regularity of the Hausdorff measure [Fed69, 2.10.2(1)] we further deduce that there exists  $C = C(n, m, F, V, U, \delta) > 1$  such that for any Borel set  $A \subseteq \{x \in U : \text{dist}(x, \mathbf{R}^n \sim U) > \delta\}$  we have*

$$C^{-1} \mathcal{H}^m(A \cap \text{spt } \|V\|) \leq \|V\|(A) \leq C \mathcal{H}^m(A \cap \text{spt } \|V\|).$$

## 10 Rectifiability of the support of the limit varifold

In 10.1 we prove that the support  $\text{spt } \|V\|$  of a  $\Phi_F$  minimising varifold  $V$  must be  $(\mathcal{H}^m, m)$  rectifiable inside any compact set  $K \subseteq U$ . Using the density ratio bounds 9.4 we also conclude, in 10.3, that the approximate tangent cones of  $\|V\|$  coincide with the classical tangent cones of  $\text{spt } \|V\|$  for *all* points  $x \in \text{spt } \|V\| \subseteq U$ . In consequence, the cones  $\text{Tan}(\text{spt } \|V\|, x)$  are in fact  $m$ -planes for  $\mathcal{H}^m$  almost all  $x \in \text{spt } \|V\|$ .

In the proof of 10.1 we follow the guidelines presented in [Alm68, 2.9(b4), p. 341]. We use only boundedness of  $F$  and make *no use* of ellipticity of  $F$ . The proof is done by contradiction. We assume that  $\text{spt } \|V\|$  is not countably  $(\mathcal{H}^m, m)$  rectifiable and we look at a density point  $x_0 \in U$  of the unrectifiable part of  $\text{spt } \|V\|$ . We choose a scale  $\rho_1 > 0$  so that the  $\mathcal{H}^m$  measure of the rectifiable part of  $\text{spt } \|V\| \cap \mathbf{B}(x_0, \rho_1)$  is negligible in comparison to the  $\mathcal{H}^m$  measure of the unrectifiable part. Then we use the deformation theorem 7.13 to produce a smooth map  $\phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$  which deforms  $\text{spt } \|V\| \cap \mathbf{B}(x_0, \rho_1)$  onto an  $m$ -dimensional skeleton of some cubical complex. Next, we apply a perturbation

argument 4.3 to obtain a map  $g$  which almost kills the  $\mathcal{H}^m$  measure of the unrectifiable part of  $\text{spt } \|V\|$  keeping the  $\mathcal{H}^m$  measure of the rectifiable part negligible.

At this point we know that  $\mathcal{H}^m(g[\text{spt } \|V\| \cap \mathbf{B}(x_0, \rho_1)])$  is significantly smaller than  $\mathcal{H}^m(\text{spt } \|V\| \cap \mathbf{B}(x_0, \rho_1))$  and we want to contradict minimality of  $V$  but we do not know whether  $\mathbf{v}(\text{spt } \|V\|) = V$ , i.e., whether  $\Phi_F(V) = \Phi_F(\text{spt } \|V\|)$ . Thus, we look at  $g[S_i]$ , where  $S_i \in \mathcal{C}$  is an appropriate minimising sequence (we need to assume it converges in the Hausdorff metric to some compact set  $S \subseteq \mathbf{R}^n$  such that  $\mathcal{H}^m(S \cap U \sim \text{spt } \|V\|) = 0$ ; see 11.2). To compare the measures of  $g[\text{spt } \|V\| \cap \mathbf{B}(x_0, \rho_1)]$  and  $g[S_i \cap \mathbf{B}(x_0, \rho_1)]$  we make use of a simple observation: these two sets both lie in the  $m$ -dimensional skeleton of a fixed cubical complex and are close in Hausdorff metric so their  $\mathcal{H}^m$  measures must also be close; see (101).

**10.1 Theorem.** *Assume*

$$\begin{aligned} U \subseteq \mathbf{R}^n \text{ is open, } \quad \mathcal{C} \text{ is a good class in } U, \quad \{S_i : i \in \mathcal{P}\} \subseteq \mathcal{C}, \quad S \in \mathcal{C}, \\ V = \lim_{i \rightarrow \infty} \mathbf{v}(S_i \cap U) \in \mathbf{V}(U), \quad F \text{ is a bounded } \mathcal{C}^0 \text{ integrand,} \\ \lim_{i \rightarrow \infty} d_{\mathcal{H}, K}(S_i \cap U, S \cap U) = 0 \quad \text{for any compact set } K \subseteq U, \\ \mathcal{H}^m(S \cap U \sim \text{spt } \|V\|) = 0, \quad \Phi_F(V) = \lim_{i \rightarrow \infty} \Phi_F(S_i \cap U) = \inf\{\Phi_F(R \cap U) : R \in \mathcal{C}\} < \infty. \end{aligned}$$

*Then  $\mathcal{H}^m(\text{spt } \|V\| \cap K) < \infty$  for any compact set  $K \subseteq U$  and  $\text{spt } \|V\|$  is a countably  $(\mathcal{H}^m, m)$  rectifiable subset of  $U$ .*

*Proof.* For  $\delta > 0$  set  $U_\delta = \{x \in U : \text{dist}(x, \mathbf{R}^n \setminus U) > \delta\}$ . Note that for each  $\delta > 0$ , by 9.4, there exists some number  $C(\delta) > 1$  such that

$$\begin{aligned} C(\delta)^{-1} \mathcal{H}^m \llcorner (\text{spt } \|V\| \cap U_\delta) &\leq \|V\| \llcorner U_\delta \leq C(\delta) \mathcal{H}^m \llcorner (\text{spt } \|V\| \cap U_\delta); \\ \text{hence, } \infty > \Phi_F(V) &\geq \inf \text{im } F \|V\|(U_\delta) \geq C(\delta)^{-1} \inf \text{im } F \mathcal{H}^m(\text{spt } \|V\| \cap U_\delta). \end{aligned}$$

This proves the first part of 10.1. We shall prove the second part by contradiction.

Assume that  $\text{spt } \|V\|$  is not countably  $(\mathcal{H}^m, m)$  rectifiable. Then there exists  $\delta > 0$  such that  $E = \text{spt } \|V\| \cap U_\delta$  is not countably  $(\mathcal{H}^m, m)$  rectifiable. We decompose  $E$  into a disjoint sum  $E = E_r \cup E_u$ , where  $E_r$  is  $(\mathcal{H}^m, m)$  rectifiable, and  $E_u$  is purely  $(\mathcal{H}^m, m)$  unrectifiable, and  $\mathcal{H}^m(E_u) > 0$ . Employing [Fed69, 2.10.19(2)(4)] we choose  $x_0 \in E_u$  such that

$$(86) \quad \Theta^m(E_r, x_0) = 0 \quad \text{and} \quad \Theta^{*m}(E_u, x_0) = b > 0.$$

Since  $F$  is bounded, there exist  $0 < C_1 < C_2 < \infty$  such that

$$(87) \quad C_1 \leq F(x, T) \leq C_2 \quad \text{for } (x, T) \in \mathbf{R}^n \times \mathbf{G}(n, m).$$

From 9.4 we see that there exist numbers  $0 < C_3 < C_4 < \infty$  (depending on  $\delta$ ) such that

$$(88) \quad C_3 \mathcal{H}^m(A) \leq \|V\|(A) \leq C_4 \mathcal{H}^m(A) \quad \text{for any Borel set } A \subset \text{spt } \|V\| \cap U_\delta.$$

Next, we fix a small number  $\varepsilon > 0$  such that

$$(89) \quad \frac{C_2(\varepsilon(1 + \varepsilon) + \varepsilon(4^m \Gamma_{7.13} + \varepsilon)(C_4 + 2))}{(1 - \varepsilon)^2 C_1 C_3} < 1.$$



Since  $\mathcal{H}^m(E) < \infty$ , we see that  $\mathcal{H}^m \llcorner E$  is a Radon measure; hence,

$$(90) \quad \mathcal{H}^m(E \cap \text{Bdry } \mathbf{B}(x_0, \rho)) > 0 \quad \text{for at most countably many } \rho > 0.$$

Employing (86) and (90) we choose  $0 < \rho_1 < \rho_3 < \rho_2$  such that

$$(91) \quad \rho_3 = (\rho_1 + \rho_2)/2, \quad \rho_1 \geq (1 - \varepsilon)^{1/m} \rho_2, \quad \mathbf{B}(x_0, \rho_2) \subseteq U_\delta,$$

$$(92) \quad \mathcal{H}^m(E \cap \text{Bdry } \mathbf{B}(x_0, \rho_i)) = 0 \quad \text{for } i = 1, 2,$$

$$(93) \quad (1 - \varepsilon)b\rho_1^m \leq \mathcal{H}^m(E_u \cap \mathbf{B}(x_0, \rho_1)) \leq (1 + \varepsilon)b\rho_1^m$$

$$(94) \quad \mathcal{H}^m(E_r \cap \mathbf{B}(x_0, \rho_1)) \leq \varepsilon b\rho_1^m, \quad \mathcal{H}^m(E \cap \mathbf{B}(x_0, \rho_2) \sim \mathbf{U}(x_0, \rho_1)) \leq \varepsilon b\rho_2^m.$$

Choose  $k \in \mathcal{P}$  such that  $2^{-k} \leq (\rho_2 - \rho_1)/8 < 2^{-k+1}$ . Define

$$\begin{aligned} \mathcal{A} &= \{Q \in \mathbf{K}_n(k) : Q \cap \mathbf{B}(x_0, \rho_3) \neq \emptyset\}, \\ \Sigma_1 &= E_r \cap \mathbf{B}(x_0, \rho_1), \quad \Sigma_2 = S, \quad \Sigma_3 = E \cap \mathbf{B}(x_0, \rho_2) \sim \mathbf{U}(x_0, \rho_1). \end{aligned}$$

Then

$$\mathbf{B}(x_0, \rho_3) \subseteq \text{Int} \bigcup \mathcal{A} \subseteq \bigcup \mathcal{A} \subseteq \mathbf{U}(x_0, \rho_2).$$

Next, apply 7.13 with  $\mathbf{K}_n(k)$ ,  $\mathcal{A}$ ,  $\Sigma_1$ ,  $\Sigma_2$ ,  $\Sigma_3$ ,  $3$ ,  $m$ ,  $m$ ,  $m$ ,  $2^{-k+8}$  in place of  $\mathcal{F}$ ,  $\mathcal{A}$ ,  $\Sigma_1$ ,  $\Sigma_2$ ,  $\Sigma_3$ ,  $l$ ,  $m_1$ ,  $m_2$ ,  $m_3$ ,  $\varepsilon$  to obtain the map  $f : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  of class  $\mathcal{C}^\infty$  called “ $g$ ” there. Define

$$A_n = \bigcup \mathcal{A}, \quad A_m = \bigcup \{Q \in \mathbf{K}_m(k) : Q \subseteq A_n\}, \quad \phi = f(1, \cdot).$$

Recalling 7.13(a)(e)(g) we see that there exists an open set  $W \subseteq \mathbf{R}^n$  such that

$$(95) \quad \begin{aligned} S &\subseteq W, \quad \phi[W \cap \mathbf{B}(x_0, \rho_3)] \subseteq A_m, \\ f(t, x) &= x \quad \text{for } x \in \mathbf{R}^n \sim \mathbf{U}(x_0, \rho_2), \end{aligned}$$

$$\int_{\Sigma_i} \|\text{D}\phi(x)\|^m d\mathcal{H}^m < \Gamma_{7.13} \mathcal{H}^m(\Sigma_i) \quad \text{for } i \in \{1, 2, 3\}.$$

For each  $\iota \in (0, 1)$  recall 7.14 and apply 4.3 with  $\mathbf{U}(x_0, \rho_3)$ ,  $E_u \cap \mathbf{B}(x_0, \rho_1)$ ,  $\phi$ ,  $\iota$  in place of  $U$ ,  $K$ ,  $f$ ,  $\varepsilon$  to get a diffeomorphism  $\varphi_\iota$  of  $\mathbf{R}^n$ . Since  $\phi$  is of class  $\mathcal{C}^\infty$ , recalling 4.5, we can find  $\iota > 0$  such that, setting

$$(96) \quad \varphi = \varphi_\iota, \quad g = \phi \circ \varphi, \quad \tau = \frac{1}{4} \min\{\varepsilon, \inf\{|x - y| : x \in S, y \in \mathbf{R}^n \sim W\}\},$$

we obtain

$$(97) \quad \varphi(x) = x \quad \text{for } x \in \mathbf{R}^n \sim \mathbf{U}(x_0, \rho_3), \quad \text{Lip}(\varphi - \text{id}_{\mathbf{R}^n}) \leq \tau,$$

$$(98) \quad |\varphi(x) - x| \leq \tau \quad \text{and} \quad \|\text{D}g(x)\|^m \leq 4^m \|\text{D}f(x)\|^m + \tau \quad \text{for } x \in \mathbf{R}^n,$$

$$(99) \quad \mathcal{H}^m(g[E_u \cap \mathbf{B}(x_0, \rho_1)]) \leq \tau \mathcal{H}^m(E_u \cap \mathbf{B}(x_0, \rho_1)).$$

Thus, employing (99), (98), (93), (94) we get

$$\begin{aligned} (100) \quad \mathcal{H}^m(g[E \cap \mathbf{B}(x_0, \rho_2)]) &\leq \tau \mathcal{H}^m(E_u \cap \mathbf{B}(x_0, \rho_1)) + \int_{\Sigma_1 \cup \Sigma_3} \|\text{D}g\|^m d\mathcal{H}^m \\ &\leq \tau(1 + \varepsilon)b\rho_1^m + 4^m \int_{\Sigma_1 \cup \Sigma_3} \|\text{D}f\|^m d\mathcal{H}^m + \tau \mathcal{H}^m(\Sigma_1 \cup \Sigma_3) \\ &\leq \tau(1 + \varepsilon)b\rho_1^m + (4^m \Gamma_{7.13} + \tau) \mathcal{H}^m(\Sigma_1 \cup \Sigma_3) \leq \tau(1 + \varepsilon)b\rho_1^m + 2\varepsilon(4^m \Gamma_{7.13} + \tau)b\rho_2^m. \end{aligned}$$

Observe that  $\varphi[S] \subseteq W$  by (96), (98). Recalling (97), (95), we see that  $g[S \cap \mathbf{B}(x_0, \rho_3)] \subseteq A_m$  so for each  $\iota > 0$  we can find an open set  $V_\iota \subseteq \mathbf{R}^n$  such that

$$(101) \quad g[S \cap \mathbf{B}(x_0, \rho_3)] \subseteq V_\iota \quad \text{and} \quad \mathcal{H}^m(V_\iota \cap A_m) \leq \mathcal{H}^m(g[S \cap \mathbf{B}(x_0, \rho_3)]) + \iota.$$

For  $\iota > 0$  set  $W_\iota = W \cap g^{-1}[V_\iota]$  and note that  $W_\iota$  is open and  $S \cap \mathbf{B}(x_0, \rho_3) \subseteq W_\iota$ . Since  $d_{\mathcal{H},K}(S_i \cap U, S \cap U) \rightarrow 0$  as  $i \rightarrow \infty$  for all compact sets  $K \subseteq U$  we see that for  $i \in \mathcal{P}$  large enough  $S_i \cap \mathbf{B}(x_0, \rho_3) \subseteq W_\iota$ ; thus,

$$\limsup_{i \rightarrow \infty} \mathcal{H}^m(g[S_i \cap \mathbf{B}(x_0, \rho_3)]) \leq \mathcal{H}^m(V_\iota \cap A_m) \leq \mathcal{H}^m(g[S \cap \mathbf{B}(x_0, \rho_3)]) + \iota.$$

But  $\iota > 0$  could be arbitrarily small, so

$$(102) \quad \limsup_{i \rightarrow \infty} \mathcal{H}^m(g[S_i \cap \mathbf{B}(x_0, \rho_3)]) \leq \mathcal{H}^m(g[S \cap \mathbf{B}(x_0, \rho_3)]).$$

Moreover, recalling that  $\mathbf{v}(S_i \cap U) \rightarrow V$  as  $i \rightarrow \infty$  and using (92) together with [All72, 2.6(2)(d)], (88), (98), and 7.13(g) we obtain

$$(103) \quad \begin{aligned} \limsup_{i \rightarrow \infty} \mathcal{H}^m(g[S_i \cap \mathbf{B}(x_0, \rho_2)] \sim \mathbf{U}(x_0, \rho_1)) \\ \leq \lim_{i \rightarrow \infty} \int_{\mathbf{B}(x_0, \rho_2) \sim \mathbf{U}(x_0, \rho_1)} \|Dg\|^m d(\mathcal{H}^m \llcorner S_i) = \int_{\mathbf{B}(x_0, \rho_2) \sim \mathbf{U}(x_0, \rho_1)} \|Dg\|^m dV \\ \leq C_4 \int_{\Sigma_3} \|Dg\|^m d\mathcal{H}^m \leq 4^m \Gamma_{7.13} C_4 \mathcal{H}^m(\Sigma_3) + C_4 \tau \mathcal{H}^m(\Sigma_3). \end{aligned}$$

Combining (102) and (103)

$$\limsup_{i \rightarrow \infty} \mathcal{H}^m(g[S_i \cap \mathbf{B}(x_0, \rho_2)]) \leq \mathcal{H}^m(g[S \cap \mathbf{B}(x_0, \rho_3)]) + (4^m \Gamma_{7.13} + \tau) C_4 \mathcal{H}^m(\Sigma_3).$$

Note that  $\mathcal{H}^m(S \cap \mathbf{B}(x_0, \rho_2)) = \mathcal{H}^m(E \cap \mathbf{B}(x_0, \rho_2))$ . In consequence, using (87), (100), (94), (93), (91), and finally (87), (88), (89), we get

$$(104) \quad \begin{aligned} \limsup_{i \rightarrow \infty} \Phi_F(g[S_i \cap \mathbf{B}(x_0, \rho_2)]) \\ \leq C_2 \mathcal{H}^m(g[E \cap \mathbf{B}(x_0, \rho_2)]) + (4^m \Gamma_{7.13} + \tau) C_2 C_4 \mathcal{H}^m(\Sigma_3) \\ \leq \gamma(1 - \varepsilon) C_1 C_3 b \rho_1^m \leq \gamma C_1 C_3 \mathcal{H}^m(S \cap \mathbf{B}(x_0, \rho_2)) \leq \gamma \Phi_F(V \llcorner \mathbf{B}(x_0, \rho_2) \times \mathbf{G}(n, m)), \end{aligned}$$

where

$$(105) \quad \gamma = \frac{C_2(\tau(1 + \varepsilon) + \varepsilon(4^m \Gamma_{7.13} + \tau)(C_4 + 2))}{(1 - \varepsilon)^2 C_1 C_3} < 1.$$

Recall that  $g(x) = x$  for  $x \in \mathbf{R}^n \sim \mathbf{U}(x_0, \rho_2)$  and  $g[\mathbf{B}(x_0, \rho_2)] \subseteq \mathbf{B}(x_0, \rho_2)$  and  $\mathbf{v}(S_i \cap U) \rightarrow V$  as  $i \rightarrow \infty$ . Hence, using (104), (105), and (92) together with (88), we obtain

$$(106) \quad \begin{aligned} \limsup_{i \rightarrow \infty} \Phi_F(g[S_i] \cap U) &= \lim_{i \rightarrow \infty} \Phi_F(S_i \cap U \sim \mathbf{B}(x_0, \rho_2)) + \limsup_{i \rightarrow \infty} \Phi_F(g[S_i \cap \mathbf{B}(x_0, \rho_2)]) \\ &< \Phi_F(V \llcorner (U \sim \mathbf{B}(x_0, \rho_2)) \times \mathbf{G}(n, m)) + \Phi_F(V \llcorner \mathbf{B}(x_0, \rho_2) \times \mathbf{G}(n, m)) = \Phi_F(V). \end{aligned}$$

Clearly  $g \in \mathfrak{D}(U)$  so  $g[S_i] \in \mathcal{C}$  for each  $i \in \mathcal{P}$  but (106) yields

$$\Phi_F(g[S_i] \cap U) < \Phi_F(V) \quad \text{for large enough } i \in \mathcal{P},$$

which contradicts  $\Phi_F(V) = \inf\{\Phi_F(R \cap U) : R \in \mathcal{C}\}$ .  $\square$

**10.2 Lemma.** *Let  $\mu$  be Radon measure over  $\mathbf{R}^n$  and  $a \in \mathbf{R}^n$ . Assume there exist  $C \in (0, \infty)$  and  $r_0 \in (0, \infty)$  such that for all  $x \in \mathbf{B}(a, r_0)$  and  $r \in (0, r_0)$*

$$\mu(\mathbf{B}(x, r)) \geq Cr^m.$$

*Then  $\text{Tan}^m(\mu, a) = \text{Tan}(\text{spt } \mu, a)$ , i.e., the approximate tangent cone of  $\mu$  at  $a$  equals the classical tangent cone of the support of  $\mu$  at  $a$ .*

*Proof.* Following [Fed69, 3.2.16] if  $a \in \mathbf{R}^n$ , and  $\varepsilon \in (0, 1)$ , and  $v \in \mathbf{R}^n$ , then we define the cone

$$(107) \quad \mathbf{E}(a, v, \varepsilon) = \{x \in \mathbf{R}^n : \exists r > 0 \ |r(x - a) - v| < \varepsilon\}.$$

Notice that if  $|v| < \varepsilon$ , then  $\mathbf{E}(a, v, \varepsilon) = \mathbf{R}^n$  and if  $0 < \varepsilon \leq |v|$ , then we may set

$$r = \frac{(x - a)}{|x - a|^2} \bullet v \quad \text{in (107)}.$$

Let  $S = \text{spt } \mu$ . By definition (see [Fed69, 3.2.16, 3.1.21]) we have

$$(108) \quad \begin{aligned} v \in \text{Tan}^m(\mu, a) &\iff \forall \varepsilon > 0 \ \Theta^{*m}(\mu \llcorner \mathbf{E}(a, v, \varepsilon), a) > 0 \\ \text{and } v \in \text{Tan}(S, a) &\iff \forall \varepsilon > 0 \ S \cap \mathbf{E}(a, v, \varepsilon) \cap \mathbf{U}(a, \varepsilon) \neq \emptyset. \end{aligned}$$

Clearly  $\text{Tan}^m(\mu, a) \subseteq \text{Tan}(S, a)$  so we only need to show the reverse inclusion. Let  $v \in \text{Tan}(S, a)$  and  $\varepsilon \in (0, 1)$  be such that  $\varepsilon \leq |v|$ . From (108) we see that there exists a sequence  $\{x_k \in \mathbf{R}^n : k \in \mathcal{P}\}$  such that

$$x_k \in S \cap \mathbf{E}(a, v, 1/k) \cap \mathbf{U}(a, 1/k).$$

Let us set  $r_k = |x_k - a|$  for  $k \in \mathcal{P}$ . Observe that whenever  $1/k < \min\{r_0, \varepsilon\}/2$ , and  $r_k < \varepsilon/2$ , and  $z \in \mathbf{B}(x_k, r_k \varepsilon/2)$ , then setting  $s = (x_k - a) \bullet v |x_k - a|^{-2}$  we obtain

$$\begin{aligned} |s(z - a) - v| &\leq |s(x_k - a) - v| + |s(z - x_k)| < \varepsilon; \\ \text{hence, } \mathbf{B}(x_k, r_k \varepsilon/2) &\subseteq \mathbf{E}(a, v, \varepsilon) \cap \mathbf{U}(a, 2r_k). \end{aligned}$$

Therefore,

$$\begin{aligned} \Theta^{*m}(\mu \llcorner \mathbf{E}(a, v, \varepsilon), a) &\geq \lim_{k \rightarrow \infty} (2r_k)^{-m} \mu(\mathbf{E}(a, v, \varepsilon) \cap \mathbf{B}(a, 2r_k)) \\ &\geq \lim_{k \rightarrow \infty} (2r_k)^{-m} \mu(\mathbf{B}(x_k, r_k \varepsilon/2)) \geq C 2^{-m} (\varepsilon/2)^m > 0. \end{aligned}$$

Since  $0 < \varepsilon \leq |v|$  could be chosen arbitrarily, we see that  $v \in \text{Tan}^m(\mu, a)$ .  $\square$

**10.3 Remark.** Let  $U$  and  $V$  be as in 10.1 and set  $E = \text{spt } \|V\| \subseteq U$ . Then for each  $a \in E$  one can find  $0 < r_0 < \text{dist}(a, \mathbf{R}^n \setminus U)$  such that  $\|V\| \llcorner \mathbf{B}(a, r_0)$  satisfies the conditions of 10.2 at  $a$ ; thus, for all points  $a \in E$  we have  $\text{Tan}^m(\|V\|, a) = \text{Tan}(E, a)$ . In particular,  $\text{Tan}(\text{spt } \|V\|, x) \in \mathbf{G}(n, m)$  for  $\mathcal{H}^m$  almost all  $x \in \text{spt } \|V\| \subseteq U$ ; see [Fed69, 3.2.19].

**10.4 Lemma.** *Let  $G \subseteq \mathbf{R}^n$  be open and bounded,  $S \subseteq \mathbf{R}^n$  be closed with  $\mathcal{H}^m(S \cap G) < \infty$  and such that  $\mathcal{H}^m(S \cap \text{Bdry } G) = 0$ , and let  $\varepsilon > 0$ . Decompose  $S \cap G$  into a disjoint sum  $S \cap G = S_u \cup S_r$ , where  $S_u$  is purely  $(\mathcal{H}^m, m)$  unrectifiable and  $S_r$  is  $(\mathcal{H}^m, m)$  rectifiable.*

*There exists a number  $\Gamma = \Gamma(n, m) \geq 1$  and a  $\mathcal{C}^\infty$  smooth map  $g \in \mathcal{D}(G)$  such that*

$$\mathcal{H}^m(g[S_u]) \leq \varepsilon \mathcal{H}^m(S_u) \quad \text{and} \quad \mathcal{H}^m(g[S_r]) \leq \Gamma \mathcal{H}^m(S_r).$$

*Proof.* Let  $\mathcal{F} = \mathbf{WF}(G)$  be the Whitney family associated to  $G$ . If  $\mathcal{H}^m(S_u) > 0$ , set  $M = \mathcal{H}^m(S_u)$  and if  $\mathcal{H}^m(S_u) = 0$ , set  $M = 1$ . Choose  $N \in \mathcal{P}$  so that

$$\mathcal{H}^m(\{x \in S : \text{dist}(x, \mathbf{R}^n \sim G) < 2^{-N}\}) < 2^{-100}\varepsilon M.$$

Define

$$\mathcal{A} = \{A \in \mathcal{F} : \mathbf{l}(Q) \geq 2^{-N-10}, \tilde{Q} \cap S \neq \emptyset\},$$

where  $\tilde{Q} = \bigcup\{R \in \mathcal{F} : R \cap Q \neq \emptyset\}$  for  $Q \in \mathcal{F}$ . In particular, we obtain

$$(109) \quad \begin{aligned} S \cap G \sim \text{Int} \bigcup \mathcal{A} &\subseteq \{x \in S : \text{dist}(x, \mathbf{R}^n \sim G) < 2^{-N}\} \\ \text{and } \mathcal{H}^m(S \cap G \sim \text{Int} \bigcup \mathcal{A}) &< 2^{-100}\varepsilon M. \end{aligned}$$

Apply the deformation theorem 7.13 with  $\mathcal{F}, \mathcal{A}, S_r, 1, m, 2^{-N-30}$  in place of  $\mathcal{F}, \mathcal{A}, \Sigma_1, l, m, \varepsilon$  to obtain the map  $f \in \mathcal{D}(G)$  of class  $\mathcal{C}^\infty$  called “ $g$ ” there. Let  $\omega$  be the modulus of continuity of  $\|Df\|$  as defined in (19) and find  $\bar{\varepsilon} > 0$  such that  $\omega(\bar{\varepsilon}) \leq 1$  and  $\bar{\varepsilon} < 2^{-100}\varepsilon$ . Next, recall 7.14 to apply the perturbation lemma 4.3 with  $S_u, f, \text{Int} \bigcup \mathcal{A}, \bar{\varepsilon}$  in place of  $K, f, U, \varepsilon$  and obtain the map  $\rho$  called “ $\rho_\varepsilon$ ” there. Set  $g = f \circ \rho$  and  $A = \bigcup \mathcal{A} + \mathbf{B}(0, 2^{-N-30})$  and  $\Gamma = 2^{2m-1}(\Gamma_{7.13} + 1)$ . To estimate  $\mathcal{H}^m(g[S_r])$  we employ 7.12 and 4.5

$$\begin{aligned} \mathcal{H}^m(g[S_r]) &\leq \mathcal{H}^m(S_r \sim A) + \int_{S_r \cap A} \|Dg\|^m d\mathcal{H}^m \\ &\leq \mathcal{H}^m(S_r \sim A) + 2^{2m-1} \int_{S_r \cap A} \|Df\|^m d\mathcal{H}^m + 2^{2m-1} \mathcal{H}^m(S_r \cap A) \\ &\leq \mathcal{H}^m(S_r \sim A) + 2^{2m-1}(\Gamma_{7.13} + 1) \mathcal{H}^m(S_r \cap A) \leq \Gamma \mathcal{H}^m(S_r). \end{aligned}$$

To estimate  $\mathcal{H}^m(g[S_u])$  we employ 4.3 and (109)

$$\begin{aligned} \mathcal{H}^m(g[S_u]) &\leq \mathcal{H}^m(S_u \sim \text{Int} \bigcup \mathcal{A}) + \bar{\varepsilon} \mathcal{H}^m(S_u \cap \text{Int} \bigcup \mathcal{A}) \\ &\leq 2^{-100}\varepsilon \mathcal{H}^m(S_u) + 2^{-100}\varepsilon \mathcal{H}^m(S_u \cap \text{Int} \bigcup \mathcal{A}) \leq \varepsilon \mathcal{H}^m(S_u) \quad \square \end{aligned}$$

## 11 Unit density of the limit varifold

In this section we finish the proof of 3.16.

We first prove a general “hair-combing” lemma 11.1, which allows to choose a sequence of sets  $\{S_i : i \in \mathcal{P}\}$  such that  $\mathbf{v}(S_i \cap U) \rightarrow V \in \mathbf{V}_m(U)$  and, additionally,  $S_i$  converge locally in Hausdorff metric to some set  $S$  such that  $\mathcal{H}^m(S \cap U \sim \text{spt} \|V\|) = 0$ . This is achieved by using first the deformation theorem 7.13 inside Whitney type cubes covering  $U \sim \text{spt} \|V\|$  and then applying the Blaschke Selection Theorem. Since the deformed sets lie in a fixed grid of cubes we know that the “hair” does not accumulate in the limit.

After that, we basically follow the guidelines presented in [Alm68, 3.2(d), p. 348 paragraph starting with “We now verify that...”, p. 349 l. 10–12] to prove that  $\text{VarTan}(V, x) = \{\mathbf{v}(\text{Tan}(\text{spt} \|V\|, x))\}$  for each  $x \in \text{spt} \|V\| \subseteq U$  such that  $\text{Tan}(\text{spt} \|V\|, x) \in \mathbf{G}(n, m)$  is an  $m$ -plane and  $\Theta^m(\|V\|, x)$  exists and is finite. That means, we take a sequence of radii  $r_j$  converging to 0 and look at the blow-up limit  $(\mu_{1/r_j})_\# V$ . At each scale we use a smooth deformation to project the part of  $V$  inside  $\mathbf{B}(x, r_j)$  onto  $x + \text{Tan}(\text{spt} \|V\|, x)$ . Then we use ellipticity of  $F$  and minimality of  $V$  to compare  $V$  with the deformed  $V$  and conclude that

$\Theta^m(\|V\|, x) = 1$ . Of course we cannot actually work with  $V$  itself but we need to always look at the minimising sequence, because  $\text{spt } \|V\|$  might not be a member of the good class  $\mathcal{C}$ . After proving that  $\Theta^m(\|V\|, x) = 1$  we still need to show that  $T = \text{Tan}(\text{spt } \|V\|, x)$  for  $V$  almost all  $(x, T) \in U \times \mathbf{G}(n, m)$ . To this end we employ the area formula and a well known relation between the tilt-excess and the measure-excess (see 11.3). Since the measure-excess vanishes in the limit, so does the tilt-excess and the theorem is proven.

We emphasize that 11.4 is the only place in the whole paper where we make use of ellipticity of  $F$ .

**11.1 Lemma.** *Let  $U \subseteq \mathbf{R}^n$  be open. Assume  $\{S_i \subseteq \mathbf{R}^n : i \in \mathcal{P}\}$  is a sequence of closed sets such that  $\mathcal{H}^m(S_i \cap U) < \infty$  for  $i \in \mathcal{P}$  and there exists a limit  $V = \lim_{i \rightarrow \infty} \mathbf{v}(S_i \cap U) \in \mathbf{V}_m(U)$ .*

*Then there exist a closed set  $X \subseteq \mathbf{R}^n$ , and  $g_i \in \mathcal{D}(U)$ , and a subsequence  $\{S'_i : i \in \mathcal{P}\}$  of  $\{S_i : i \in \mathcal{P}\}$  such that, setting  $E = \text{spt } \|V\| \cup (\mathbf{R}^n \sim U)$  and  $X_i = g_i[S'_i]$  for  $i \in \mathcal{P}$ , we obtain*

$$\begin{aligned} \lim_{i \rightarrow \infty} d_{\mathcal{H}, K}(X_i \cap U, X \cap U) &= 0 \quad \text{for each compact set } K \subseteq U, \\ \lim_{i \rightarrow \infty} \sup \{r \in \mathbf{R} : \mathcal{H}^m(\{x \in X_i \cap K : \text{dist}(x, E) \geq r\}) > 0\} &= 0 \quad \text{for } K \subseteq \mathbf{R}^n \text{ compact}, \\ \lim_{i \rightarrow \infty} \mathbf{v}(X_i \cap U) &= V, \quad \mathcal{H}^m(X \cap U \sim \text{spt } \|V\|) = 0. \end{aligned}$$

*Furthermore, if  $E$  is bounded and  $S_i$  is compact for each  $i \in \mathcal{P}$  and  $\sup\{\mathcal{H}^m(S_i \cap U) : i \in \mathcal{P}\} < \infty$ , then*

$$\sup\{\text{diam } X_i : i \in \mathcal{P}\} < \infty.$$

*Proof.* Let  $\mathcal{F} = \{Q_j : j \in \mathcal{P}\} = \mathbf{WF}(U \sim \text{spt } \|V\|)$  be the Whitney family defined in 7.5. For brevity of the notation, if  $Q \in \mathcal{F}$ , we define

$$\mathbf{N}(Q, 0) = \{Q\}, \quad \mathbf{N}(Q, i) = \{R \in \mathcal{F} : R \cap \bigcup \mathbf{N}(Q, i-1) \neq \emptyset\} \quad \text{for } i = 1, 2, \dots,$$

and we set  $\tilde{Q} = \bigcup \mathbf{N}(Q, 1)$ . Since  $\mathbf{v}(S_i \cap U) \rightarrow V$  in  $\mathbf{V}_m(U)$  as  $i \rightarrow \infty$ , using [All72, 2.6.2(c)], we see that

$$\limsup_{i \rightarrow \infty} \mathcal{H}^m(S_i \cap \bigcup \mathbf{N}(Q_j, 3)) \leq \|V\|(\bigcup \mathbf{N}(Q_j, 3)) = 0 \quad \text{for each } j \in \mathcal{P}.$$

Set  $S_i^0 = S_i$  for  $i \in \mathcal{P}$  and define inductively  $S_i^j$  for  $j \in \mathcal{P}$  by requiring that  $\{S_i^j : i \in \mathcal{P}\}$  be a subsequence of  $\{S_i^{j-1} : i \in \mathcal{P}\}$  satisfying

$$(110) \quad \mathcal{H}^m(S_i^j \cap \bigcup \mathbf{N}(Q_j, 3)) < \frac{l(Q_j)^m}{2^{m+i}\Gamma_{7.13}} \quad \text{for } i \in \mathcal{P}.$$

For  $j \in \mathcal{P}$  define  $P_j = S_j^j$  and  $\mathcal{A}_j \subseteq \mathcal{F}$  to consist of all the cubes  $Q \in \mathcal{F}$  with  $Q \subseteq \mathbf{B}(0, 2^j)$  and satisfying

$$\begin{aligned} \text{either } P_j \cap \tilde{Q} \neq \emptyset \text{ and } \mathcal{H}^m(P_j \cap \bigcup \mathbf{N}(Q, 3)) &< \frac{l(Q)^m}{2^{m+j}\Gamma_{7.13}} \text{ and } l(Q) \geq \frac{1}{2^j} \\ \text{or } P_j \cap \tilde{Q} \neq \emptyset \text{ and } l(Q) &\geq 1. \end{aligned}$$

Clearly  $\mathcal{A}_j$  are finite. For each  $j \in \mathcal{P}$  apply 7.13 with  $\mathcal{F}$ ,  $\mathcal{A}_j$ ,  $P_j$ ,  $m$ ,  $1$ ,  $2^{-j-8}$  in place of  $\mathcal{F}$ ,  $\mathcal{A}$ ,  $\Sigma_1$ ,  $m_1$ ,  $l$ ,  $\varepsilon$  to obtain the map  $\bar{f}_j \in \mathcal{C}^\infty(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$  called “ $f$ ” there. Set

$$f_j = \bar{f}_j(1, \cdot) \in \mathcal{D}(U) \quad \text{and} \quad W_j = f_j[P_j].$$

We shall prove that  $\lim_{j \rightarrow \infty} \mathbf{v}(W_j \cap U) = V$  in  $\mathbf{V}_m(U)$ .

Let  $\varphi \in \mathcal{H}(U \times \mathbf{G}(n, m))$  and for  $j \in \mathcal{P}$  let  $\zeta_j \in \mathcal{C}^\infty(\mathbf{R}^n, [0, 1])$  be such that

$$\zeta_j(x) = 1 \quad \text{if } \text{dist}(x, \bigcup \mathcal{A}_j) \leq 2^{-j-4}, \quad \zeta_j(x) = 0 \quad \text{if } \text{dist}(x, \bigcup \mathcal{A}_j) \geq 2^{-j-2}.$$

This choice ensures

$$\begin{aligned} \text{spt}(\mathbf{1}_{\mathbf{R}^n} - \zeta_j) &\subseteq \{x \in \mathbf{R}^n : \text{dist}(x, \bigcup \mathcal{A}_j) > 2^{-j-6}\} \subseteq \{x \in \mathbf{R}^n : f_j(x) = x\}, \\ \text{spt } \zeta_j &\subseteq \bigcup \{\tilde{Q} : Q \in \mathcal{A}_j\} \subseteq U. \end{aligned}$$

We set  $\bar{\zeta}_j(x, T) = \zeta_j(x)$  for  $(x, T) \in \mathbf{R}^n \times \mathbf{G}(n, m)$  and then

$$\begin{aligned} \mathbf{v}(W_j \cap U)(\varphi) &= \mathbf{v}(W_j \cap U)(\varphi \bar{\zeta}_j) + \mathbf{v}(P_j \cap U)(\varphi(\mathbf{1}_{\mathbf{R}^n \times \mathbf{G}(n, m)} - \bar{\zeta}_j)) \\ &= \mathbf{v}(P_j \cap U)(\varphi) + \mathbf{v}(W_j \cap U)(\varphi \bar{\zeta}_j) - \mathbf{v}(P_j \cap U)(\varphi \bar{\zeta}_j). \end{aligned}$$

Since  $\{P_j : j \in \mathcal{P}\}$  is a subsequence of  $\{S_j : j \in \mathcal{P}\}$  and  $\mathbf{v}(S_j \cap U) \rightarrow V$  in  $\mathbf{V}_m(U)$  as  $j \rightarrow \infty$ , we only need to show that  $\lim_{j \rightarrow \infty} \mathbf{v}(W_j \cap U)(\varphi \bar{\zeta}_j) = 0$  and  $\lim_{j \rightarrow \infty} \mathbf{v}(P_j \cap U)(\varphi \bar{\zeta}_j) = 0$ . Set

$$\mathcal{G} = \{Q \in \mathcal{F} : \tilde{Q} \cap \text{spt } \varphi \neq \emptyset, \mathbf{l}(Q) < 1\} \quad \text{and} \quad \mathcal{J} = \{Q \in \mathcal{F} : \tilde{Q} \cap \text{spt } \varphi \neq \emptyset, \mathbf{l}(Q) \geq 1\},$$

then  $\mathcal{G}$  and  $\mathcal{J}$  are finite and do not depend on  $j \in \mathcal{P}$ . Moreover,  $\text{spt } \varphi \cap \text{spt } \zeta_j \subseteq \bigcup \{\tilde{Q} : Q \in \mathcal{A}_j \cap \mathcal{G}\} \cup \bigcup \{\tilde{Q} : Q \in \mathcal{J}\}$ , and  $\text{dist}(\bigcup \{\tilde{Q} : Q \in \mathcal{J}\}, \text{spt } \|V\|) > 0$ ; hence,

$$\begin{aligned} (111) \quad \lim_{j \rightarrow \infty} \mathbf{v}(P_j \cap U)(\varphi \bar{\zeta}_j) &\leq \lim_{j \rightarrow \infty} \sup \text{im } |\varphi| \mathcal{H}^m(\text{spt } \varphi \cap \text{spt } \zeta_j \cap P_j) \\ &\leq \sup \text{im } |\varphi| \lim_{j \rightarrow \infty} \left( \sum_{Q \in \mathcal{G} \cap \mathcal{A}_j} \mathcal{H}^m(P_j \cap \tilde{Q}) + \sum_{Q \in \mathcal{J}} \mathcal{H}^m(P_j \cap \tilde{Q}) \right) \\ &\leq \sup \text{im } |\varphi| \left( \lim_{j \rightarrow \infty} \mathcal{H}^0(\mathcal{G}) \Gamma_{7.13}^{-1} 2^{-j} + \sum_{Q \in \mathcal{J}} \lim_{j \rightarrow \infty} \mathcal{H}^m(P_j \cap \tilde{Q}) \right) = 0. \end{aligned}$$

To deal with  $\mathbf{v}(W_j \cap U)(\varphi \bar{\zeta}_j)$  we first note that whenever  $Q \in \mathcal{F}$ , then, recalling 7.13(g),

$$\begin{aligned} (112) \quad \mathcal{H}^m(f_j[P_j] \cap \tilde{Q}) &\leq \sum_{R \in \mathbf{N}(Q, 1)} \mathcal{H}^m(f_j[P_j] \cap R) \\ &\leq \sum_{R \in \mathbf{N}(Q, 1)} \Gamma_{7.13} \mathcal{H}^m(P_j \cap (\tilde{R} + \mathbf{U}(0, 2^{-j-8}))) \leq \Delta^2 \Gamma_{7.13} \mathcal{H}^m(P_j \cap \bigcup \mathbf{N}(Q, 3)), \end{aligned}$$

where  $\Delta = \Delta(n) \in \mathcal{P}$  be so big that  $\Delta \geq \mathcal{H}^0(\mathbf{N}(T, 1))$  for all  $T \in \mathcal{F}$ . Now, we can estimate as in (111) using (112)

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathbf{v}(W_j \cap U)(\varphi \bar{\zeta}_j) &\leq \sup \text{im } |\varphi| \lim_{j \rightarrow \infty} \left( \sum_{Q \in \mathcal{G} \cap \mathcal{A}_j} \mathcal{H}^m(f_j[P_j] \cap \tilde{Q}) + \sum_{Q \in \mathcal{J}} \mathcal{H}^m(f_j[P_j] \cap \tilde{Q}) \right) \\ &\leq \sup \text{im } |\varphi| \left( \lim_{j \rightarrow \infty} \mathcal{H}^0(\mathcal{G}) \Delta^2 2^{-j} + \Delta^2 \Gamma_{7.13} \sum_{Q \in \mathcal{J}} \lim_{j \rightarrow \infty} \mathcal{H}^m(P_j \cap \bigcup \mathbf{N}(Q, 3)) \right) = 0. \end{aligned}$$

This finishes the proof that  $\lim_{j \rightarrow \infty} \mathbf{v}(W_j \cap U) = V$ .

Let  $\{K_i \subseteq U : i \in \mathcal{P}\}$  be a sequence of compact sets such that  $\bigcup \{K_i \subseteq U : i \in \mathcal{P}\} = U$  and  $K_i \subseteq K_{i+1}$  for  $j \in \mathcal{P}$ . We set  $W_j^0 = W_j$  for  $j \in \mathcal{P}$ . For  $i \in \mathcal{P}$  we define  $\{W_j^i : j \in \mathcal{P}\}$  to be a subsequence of  $\{W_j^{i-1} : j \in \mathcal{P}\}$  such that the sequence  $\{W_j^i \cap K_i : j \in \mathcal{P}\}$  converges in the Hausdorff metric to some compact set  $F_i$  – this can be done employing the Blaschke Selection Theorem; see [Pri40]. Finally, we set  $R_i = W_i^i$  for  $i \in \mathcal{P}$  and  $R = \bigcup \{F_i : i \in \mathcal{P}\}$ . We see that for any compact set  $K \subseteq U$  we have  $d_{\mathcal{H},K}(R_i \cap U, R \cap U) \rightarrow 0$  as  $i \rightarrow \infty$ .

Now, we need to show that  $\mathcal{H}^m(R \cap U \sim \text{spt } \|V\|) = 0$ . It will be enough to prove that  $\mathcal{H}^m(R \cap Q) = 0$  for each  $Q \in \mathcal{F}$ . Recall that  $R \cap Q$  is the limit, in the Hausdorff metric, of some subsequence of  $\{f_j[P_j] \cap Q : j \in \mathcal{P}\}$ . We claim that for big enough  $j \in \mathcal{P}$  the set  $f_j[P_j] \cap Q$  lies in the  $m-1$ -dimensional skeleton of  $\mathcal{F}$ , i.e.,  $f_j[P_j] \cap Q \subseteq \bigcup \mathbf{K}_{m-1} \cap \mathbf{CX}(\mathcal{F})$ , which implies that  $R \cap Q \subseteq \bigcup \mathbf{K}_{m-1} \cap \mathbf{CX}(\mathcal{F})$  and  $\mathcal{H}^m(R \cap Q) = 0$ . To prove our claim let  $j_0, j_1, j_2 \in \mathbf{Z}$  be such that  $Q = Q_{j_0}$  and  $\mathbf{l}(Q) = 2^{-j_1}$  and  $Q \subseteq \mathbf{B}(0, 2^{j_2})$ , and assume  $j > \max\{j_0, j_1, j_2\}$ .

In case  $\tilde{Q} \cap P_j = \emptyset$  we have  $f_j[P_j] \cap Q = \emptyset$ , by 7.13(d), and there is nothing to prove. Thus, we assume that  $\tilde{Q} \cap P_j \neq \emptyset$  which implies that  $Q \in \mathcal{A}_j$  due to (110) and  $j > \max\{j_0, j_1, j_2\}$ . For  $R \in \mathbf{N}(Q, 1)$  such that  $R \cap P_j \neq \emptyset$  we estimate using 7.13(g)

$$(113) \quad \mathcal{H}^m(f_j[P_j] \cap R) \leq \Gamma_{7.13} \mathcal{H}^m(P_j \cap (\tilde{R} + \mathbf{U}(0, 2^{-j-8}))) \\ \leq \Gamma_{7.13} \mathcal{H}^m(P_j \cap \bigcup \mathbf{N}(Q, 3)) < 2^{-m} \mathbf{l}(Q)^m \leq \mathbf{l}(R)^m.$$

If  $R \cap P_j \neq \emptyset$  and  $R \in \mathbf{N}(Q, 1)$ , then it follows that  $R \subseteq \text{Int } \bigcup \mathcal{A}_j$ ; hence, combining (113) with 7.13(f), we see that  $f_j[P_j] \cap R \subseteq \bigcup \mathbf{K}_{m-1} \cap \mathbf{CX}(\mathcal{F})$ . Next, observe that

$$(114) \quad f_j[P_j] \cap Q \subseteq \bigcup \{f_j[R] : R \in \mathbf{N}(Q, 1), R \cap P_j \neq \emptyset\} \subseteq \bigcup \mathbf{K}_{m-1} \cap \mathbf{CX}(\mathcal{F}).$$

Let  $K \subseteq \mathbf{R}^n$  be compact. Observe that for each  $k \in \mathcal{P}$ , there are only finitely many cubes in  $\mathcal{F}$  which touch  $K$  and have side length at least  $2^{-k}$ . If  $j_0 \in \mathcal{P}$  is the maximal index of such cube and  $j_1 \in \mathcal{P}$  is such that  $Q_{j_0} \subseteq \mathbf{B}(0, 2^{j_1})$ , then for  $j > \max\{j_0, j_1, k\}$  the estimate (113) holds for any  $R \in \mathbf{N}(Q, 1)$  whenever  $Q \in \mathcal{F}$  satisfies  $\mathbf{l}(Q) \geq 2^{-k}$  and  $Q \cap K \neq \emptyset$ . In consequence, as in (114),

$$f_j[P_j] \cap \bigcup \{Q \in \mathcal{F} : \mathbf{l}(Q) \geq 2^{-k}, Q \cap K \neq \emptyset\} \subseteq \bigcup \mathbf{CX}(\mathcal{F}) \cap \mathbf{K}_{m-1}.$$

Recalling that  $\mathcal{F} = \mathbf{WF}(U \sim \text{spt } \|V\|)$  was the Whitney family associated to  $\mathbf{R}^n \sim E$ , we see that there exists  $\{r_j \in (0, \infty) : j \in \mathcal{P}\}$  such that  $r_j \downarrow 0$  as  $j \rightarrow \infty$  and

$$\mathcal{H}^m(\{x \in f_j[P_j] \cap K : \text{dist}(x, E) \geq r_j\}) = 0 \quad \text{for } j \in \mathcal{P}.$$

Observe that for each  $i \in \mathcal{P}$  there exists  $j(i) \in \mathcal{P}$  such that  $R_i = f_{j(i)}[P_{j(i)}]$ , so setting  $S'_i = P_{j(i)}$  for  $i \in \mathcal{P}$  we obtain a subsequence of  $\{S_i : i \in \mathcal{P}\}$ . Hence, we may finish the proof of the first part of 11.1 by setting  $X = R$  and  $g_i = f_{j(i)}$  and  $X_i = R_i$  for  $i \in \mathcal{P}$ .

Assume now that  $E$  is bounded and  $S_i$  is compact for  $i \in \mathcal{P}$  and  $\sup\{\mathcal{H}^m(S_i \cap U) : i \in \mathcal{P}\} < \infty$ . In this case we need to further modify the sets  $R_i$  to ensure that all the resulting sets  $X_i$  are bounded. To this end we shall simply project the sets  $R_j$  onto the cube  $[-M_1, M_1]^n$ . Since our definition of admissible mappings allows only for deformations inside convex sets we need to perform the projection in several steps. Using the fact that outside  $[-M_1, M_1]^n$  the sets  $R_j$  lie in the  $m-1$  dimensional skeleton of  $\mathcal{F}$  we will deduce that the projected sets  $X_j$  give rise to the same varifolds as  $R_j$ , i.e.,  $\mathbf{v}(R_j \cap U) = \mathbf{v}(X_j \cap U)$ .

Suppose there exists  $M_0 > 1$  such that

$$E = \text{spt } \|V\| \cup (\mathbf{R}^n \sim U) \subseteq [-M_0, M_0]^n \quad \text{and} \quad \sup\{\mathcal{H}^m(S_i \cap U) : i \in \mathcal{P}\} < M_0.$$

Then  $\sup\{\mathcal{H}^m(R_i \cap U) : i \in \mathcal{P}\} < M_0$  and we can find  $M_1 > 2^{10}M_0$  such that

$$\bigcup\{Q \in \mathcal{F} : \mathbf{l}(Q)^m \leq \Gamma_{7.13}M_0\} \subseteq [-2^{-10}M_1, 2^{-10}M_1]^n.$$

Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbf{R}^n$ . For  $i \in \{-n, \dots, -1, 1, \dots, n\}$  and  $j \in \mathcal{P}$  we define

- $L_i = \{x \in \mathbf{R}^n : x \bullet e_{|i|} = \frac{i}{|i|}M_1\}$ ,
- $H_i = \{x \in \mathbf{R}^n : x \bullet e_{|i|} \frac{i}{|i|} \geq M_1\}$ ,
- $p_i$  to be the affine orthogonal projection onto the affine plane  $L_i$ ,
- $Y_{j,i} \subseteq H_i$  to be a large cube containing  $[-M_1, M_1]^n \cap L_i$  and such that

$$R_j \cap H_i \subseteq Y_{j,i},$$

- $\varphi_{i,j} \in \mathcal{C}^\infty(\mathbf{R}^n)$  to be a cut-off function such that  $0 \leq \varphi_{i,j}(x) \leq 1$  for  $x \in \mathbf{R}^n$  and

$$Y_{j,i} \subseteq \varphi_{j,i}^{-1}\{1\} \quad \text{and} \quad \text{spt } \varphi_{j,i} \subseteq \mathbf{R}^n \sim [-2^{-1}M_1, 2^{-1}M_1]^n \text{ is compact and convex,}$$

- $h_{j,i} = p_i \varphi_{j,i} + (\mathbb{1}_{\mathbf{R}^n} - \varphi_{j,i}) \text{id}_{\mathbf{R}^n} \in \mathfrak{D}(U)$ ,
- $h_j = h_{j,-n} \circ \dots \circ h_{j,-1} \circ h_{j,1} \circ \dots \circ h_{j,n} \in \mathfrak{D}(U)$ .

We set  $X_i = h_i[R_i]$  and  $g_i = h_i \circ f_{j(i)}$  for  $i \in \mathcal{P}$ . Clearly  $X_j \subseteq [-M_1, M_1]^n$  for each  $j \in \mathcal{P}$  so, employing the Blaschke Selection Theorem, we can assume that  $X_j$  converges in the Hausdorff metric to some compact set  $X$ . Observe that if  $i, j \in \mathcal{P}$  and  $R_i = f_j[P_j]$  and  $Q \in \mathcal{F}$  is such that  $Q \cap P_j \neq \emptyset$  and  $Q \cap \mathbf{R}^n \sim [-2^{-1}M_1, 2^{-1}M_1]^n \neq \emptyset$ , then  $\mathbf{l}(Q)^m > \Gamma_{7.13}M_0 > 1$  and  $Q \in \mathcal{A}_j$  and  $\Gamma_{7.13}\mathcal{H}^m(P_j \cap \bigcup N(Q, 3)) \leq \Gamma_{7.13}M_0 < \mathbf{l}(Q)^m$ ; hence,  $f_j[P_j] \cap Q \subseteq \mathbf{CX}(\mathcal{F}) \cap \mathbf{K}_{m-1}$ , by 7.13(f), and  $\mathcal{H}^m(f_j[P_j] \cap Q) = 0$  so  $\mathcal{H}^m(h_i[R_i] \cap Q) = 0$ . Since  $X_i \cap [-2^{-1}M_1, 2^{-1}M_1]^n = R_i \cap [-2^{-1}M_1, 2^{-1}M_1]^n$  for  $i \in \mathcal{P}$ , we see that  $\mathbf{v}(R_i \cap U) = \mathbf{v}(X_i \cap U)$  for  $i \in \mathcal{P}$  and  $\mathbf{v}(X_i \cap U) \rightarrow V \in \mathbf{V}_m(U)$  as  $i \rightarrow \infty$ .  $\square$

## 11.2 Corollary. Assume

$$U \subseteq \mathbf{R}^n \text{ is open, } F \text{ is a bounded } \mathcal{C}^0 \text{ integrand,} \\ \mathcal{C} \text{ is a good class in } U, \quad \mu = \inf\{\Phi_F(R \cap U) : R \in \mathcal{C}\} \in (0, \infty).$$

Then there exist  $\{S_i : i \in \mathcal{P}\} \subseteq \mathcal{C}$  and  $S \in \mathcal{C}$  and  $V \in \mathbf{V}_m(U)$  and  $E \subseteq \mathbf{R}^n$  such that

$$E = \text{spt } \|V\| \cup (\mathbf{R}^n \sim U), \quad \mathcal{H}^m(S \cap U \sim \text{spt } \|V\|) = 0, \\ \lim_{i \rightarrow \infty} d_{\mathcal{H},K}(S_i \cap U, S \cap U) = 0 \quad \text{for } K \subseteq U \text{ compact,} \\ \lim_{i \rightarrow \infty} \sup\{r \in \mathbf{R} : \mathcal{H}^m(\{x \in S_i \cap K : \text{dist}(x, E) \geq r\}) > 0\} = 0 \quad \text{for } K \subseteq \mathbf{R}^n \text{ compact,} \\ \sup\{\mathcal{H}^m(S_i \cap U) : i \in \mathcal{P}\} < \infty, \quad \lim_{i \rightarrow \infty} \mathbf{v}(S_i \cap U) = V, \quad \lim_{i \rightarrow \infty} \Phi_F(S_i \cap U) = \mu.$$

Furthermore, if  $B = \mathbf{R}^n \sim U$  is compact and  $S_i$  is compact for  $i \in \mathcal{P}$ , then

$$\text{spt } \|V\| \text{ is bounded and } \sup\{\text{diam } S_i : i \in \mathcal{P}\} < \infty,$$



*Proof.* Let  $\{R_i : i \in \mathcal{P}\}$  be any minimising sequence in  $\mathcal{C}$ , i.e.,

$$\Phi_F(R_i \cap U) \rightarrow \inf\{\Phi_F(R \cap U) : R \in \mathcal{C}\} = \mu \quad \text{as } i \rightarrow \infty.$$

Observe that since  $F$  is bounded and  $\mu$  is finite we have  $\mathcal{H}^m(R_i \cap U) < 2(\inf \text{im } F)^{-1}\mu$  for all but finitely many  $i \in \mathcal{P}$  – we shall assume it holds for all  $i \in \mathcal{P}$ . In consequence, we can choose a subsequence  $\{R'_i : i \in \mathcal{P}\}$  of  $\{R_i : i \in \mathcal{P}\}$  such that  $\mathbf{v}(R'_i \cap U)$  converges as  $i \rightarrow \infty$  to some  $V \in \mathbf{V}_m(U)$ . If  $B = \mathbf{R}^n \sim U$  is compact, then we use 9.4 too see that  $\text{spt } \|V\|$  must be bounded; hence,  $E$  is bounded. Next, we apply 11.1 to  $\{R'_i : i \in \mathcal{P}\}$  to obtain a subsequence  $\{P_i : i \in \mathcal{P}\}$  of  $\{R'_i : i \in \mathcal{P}\}$  and maps  $\{g_i \in \mathfrak{D}(U) : i \in \mathcal{P}\}$  and  $S \in \mathcal{C}$ . Finally, we set  $S_i = g_i[P_i]$ .  $\square$

In the next lemma we relate the  $L^2$  tilt-excess to the measure-excess; see (140).

**11.3 Lemma.** *Let  $P, Q \in \mathbf{G}(n, m)$ . Then*

$$\frac{1}{2}\|P_{\natural} - Q_{\natural}\|^2 \leq 1 - \|\bigwedge_m P_{\natural} \circ Q_{\natural}\| \leq 2^{2m+3}\|P_{\natural} - Q_{\natural}\|^2.$$

*Proof.* Employ [All72, 8.9(3)] to find  $u \in Q$  such that  $|u| = 1$  and  $\|P_{\natural} - Q_{\natural}\| = |P_{\natural}^{\perp} u|$ . Let  $u_1, \dots, u_m$  be an orthonormal basis of  $Q$  such that  $u_1 = u$ . We have

$$\|\bigwedge_m P_{\natural} \circ Q_{\natural}\| = |P_{\natural} u_1 \wedge \dots \wedge P_{\natural} u_m| \leq |P_{\natural} u_1| = (1 - \|P_{\natural} - Q_{\natural}\|^2)^{1/2}.$$

In consequence, since  $(1 - x)^{1/2} \leq 1 - \frac{1}{2}x$  for  $x \in [0, 1]$ , we obtain

$$1 - \|\bigwedge_m P_{\natural} \circ Q_{\natural}\| \geq 1 - (1 - \|P_{\natural} - Q_{\natural}\|^2)^{1/2} \geq \frac{1}{2}\|P_{\natural} - Q_{\natural}\|^2.$$

Next, we shall derive the upper estimate. Let  $q \in \mathbf{O}^*(n, m)$  be such that  $\text{im } q^* = Q$ . Since  $P_{\natural} \circ Q_{\natural} = Q_{\natural} - P_{\natural}^{\perp} \circ Q_{\natural}$  and  $q^* \circ q = Q_{\natural}$  and  $(P_{\natural}^{\perp})^* = P_{\natural}^{\perp}$  we obtain, using [Fed69, 1.7.6, 1.7.9, 1.4.5],

$$\begin{aligned} \|\bigwedge_m P_{\natural} \circ Q_{\natural}\|^2 &= |\bigwedge_m P_{\natural} \circ Q_{\natural}|^2 = |\bigwedge_m (q^* - P_{\natural}^{\perp} \circ q^*) \circ q|^2 = |\bigwedge_m (q^* - P_{\natural}^{\perp} \circ q^*)|^2 \\ &= \text{trace}(\bigwedge_m (q^* - P_{\natural}^{\perp} \circ q^*)^* \circ (q^* - P_{\natural}^{\perp} \circ q^*)) = \det(\text{id}_{\mathbf{R}^m} - q \circ P_{\natural}^{\perp} \circ q^*) \\ &= \sum_{j=0}^m (-1)^j \text{trace}(\bigwedge_j q \circ P_{\natural}^{\perp} \circ q^*) = \sum_{j=0}^m (-1)^j \text{trace}(\bigwedge_j (P_{\natural}^{\perp} \circ q^*)^* \circ (P_{\natural}^{\perp} \circ q^*)) \\ &= \sum_{j=0}^m (-1)^j |\bigwedge_j (P_{\natural}^{\perp} \circ q^*)|^2 = 1 - |P_{\natural}^{\perp} \circ Q_{\natural}|^2 + E, \end{aligned}$$

where  $E = \sum_{j=2}^m (-1)^j |\bigwedge_j (P_{\natural}^{\perp} \circ q^*)|^2$ . Note that  $1 - x \leq (1 - x)^{1/2}$  for  $x \in [0, 1]$ ; hence,

$$(115) \quad 1 - \|\bigwedge_m P_{\natural} \circ Q_{\natural}\| = 1 - (1 - |P_{\natural}^{\perp} \circ Q_{\natural}|^2 + E)^{1/2} \leq |P_{\natural}^{\perp} \circ Q_{\natural}|^2 - E.$$

Employing [Fed69, 1.7.6, 1.7.9, 1.3.2] together with [All72, 8.9(3)] we get

$$(116) \quad |\bigwedge_j (P_{\natural}^{\perp} \circ q^*)|^2 \leq \binom{m}{j} \|P_{\natural} - Q_{\natural}\|^{2j} \quad \text{for } j = 0, 1, \dots, m.$$

If  $\|P_{\natural} - Q_{\natural}\|^2 \leq 2^{-(m+2)}$ , then

$$(117) \quad |E| \leq \sum_{j=2}^m \binom{m}{j} \|P_{\natural} - Q_{\natural}\|^{2j} \leq 2^m \sum_{j=2}^m \|P_{\natural} - Q_{\natural}\|^{2j} \\ \leq 2^m \frac{\|P_{\natural} - Q_{\natural}\|^4 - \|P_{\natural} - Q_{\natural}\|^{2m}}{1 - \|P_{\natural} - Q_{\natural}\|^2} \leq \frac{1}{2} \|P_{\natural} - Q_{\natural}\|^2.$$

If  $2^{-(m+2)} < \|P_{\natural} - Q_{\natural}\|^2 = \|P_{\natural}^{\perp} \circ Q_{\natural}\| \leq 1$ , then

$$(118) \quad |E| \leq \sum_{j=2}^m \binom{m}{j} \|P_{\natural} - Q_{\natural}\|^{2j} \leq 2^m \leq 2^{2m+2} \|P_{\natural} - Q_{\natural}\|^2.$$

Combining (115), (116), (117), and (118) we obtain

$$1 - \|\bigwedge_m P_{\natural} \circ Q_{\natural}\| \leq 2^{m+3} \|P_{\natural} - Q_{\natural}\|^2. \quad \square$$

**11.4 Theorem.** *Assume*

$$\begin{aligned} & U \subseteq \mathbf{R}^n \text{ is open, } F \text{ is a bounded elliptic } \mathcal{C}^0 \text{ integrand,} \\ & \mathcal{C} \text{ is a good class in } U, \quad \{S_i : i \in \mathcal{P}\} \subseteq \mathcal{C}, \quad V = \lim_{i \rightarrow \infty} \mathbf{v}(S_i \cap U) \in \mathbf{V}(U), \\ & x_0 \in \text{spt } \|V\| \subseteq U, \quad T = \text{Tan}(\text{spt } \|V\|, x_0) \in \mathbf{G}(n, m), \quad \Theta^m(\|V\|, x_0) \in (0, \infty), \\ & \Phi_F(V) = \lim_{i \rightarrow \infty} \Phi_F(S_i \cap U) = \inf\{\Phi_F(R \cap U) : R \in \mathcal{C}\} = \mu \in (0, \infty). \end{aligned}$$

Then  $\text{VarTan}(V, x_0) = \{\mathbf{v}(T)\}$ .

*Proof.* Define  $E = \text{spt } \|V\|$  and  $B = \mathbf{R}^n \sim U$ . Without loss of generality, we assume  $x_0 = 0$ . Employing 11.2 we shall also assume that  $\{S_i : i \in \mathcal{P}\}$  satisfies all the conditions of 11.2. In particular, for some  $S \in \mathcal{C}$  and all compacts sets  $K \subseteq U$

$$(119) \quad \lim_{i \rightarrow \infty} d_{\mathcal{H}, K}(S_i \cap U, S \cap U) = 0, \\ \limsup_{i \rightarrow \infty} \{r \in \mathbf{R} : \mathcal{H}^m(\{x \in S_i \cap K : \text{dist}(x, E \cup B) \geq r\}) > 0\} = 0.$$

Define

$$\delta(r) = \sup \left\{ \frac{\text{dist}(x, T)}{|x|} : x \in E \cap \mathbf{U}(x, 2r) \sim \{0\} \right\} \quad \text{for } r \in (0, \infty).$$

Recall [All72, 3.4(1)] and  $\Theta^m(\|V\|, x_0) \in (0, \infty)$  to see that  $\text{VarTan}(V, 0)$  is compact and nonempty so we can choose  $C \in \text{VarTan}(V, 0)$  and  $\{r_j \in \mathbf{R} : j \in \mathcal{P}\}$  such that  $r_j \downarrow 0$  as  $j \rightarrow \infty$ , and  $\delta(r_1) < 1$ , and  $\mathbf{U}(0, 3r_1) \subseteq U$ , and

$$C = \lim_{j \rightarrow \infty} (\mu_{1/r_j})_{\#} V, \quad \text{and} \quad \|V\|(\text{Bdry } \mathbf{B}(0, r_j)) = 0 \quad \text{for } j \in \mathcal{P}.$$

Set  $\delta_j = \delta(r_j)$  and  $\varepsilon_j = 10\delta_j^{1/2}$ . For  $j \in \mathcal{P}$  let  $f_j, g_j, h_j \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$  be such that

$$\begin{aligned} f_j(t) &= 1 \quad \text{for } t \leq (1 - \varepsilon_j)r_j, \quad f_j(t) = 0 \quad \text{for } t \geq (1 - \varepsilon_j/2)r_j, \\ 0 &\leq f_j(t) \leq 1 \quad \text{and} \quad |f'_j(t)| \leq 4/(\varepsilon_j r_j) \quad \text{for } t \in \mathbf{R}, \\ g_j(t) &= 0 \quad \text{for } t \leq (1 - 3\varepsilon_j)r_j \text{ or } t \geq (1 - \varepsilon_j/2)r_j, \\ g_j(t) &= 1 \quad \text{for } (1 - 2\varepsilon_j)r_j \leq t \leq (1 - \varepsilon_j)r_j, \\ 0 &\leq g_j(t) \leq 1 \quad \text{and} \quad |g'_j(t)| \leq 4/(\varepsilon_j r_j) \quad \text{for } t \in \mathbf{R}, \\ h_j(t) &= 1 \quad \text{for } t \leq 2\delta_j r_j, \quad h_j(t) = 0 \quad \text{for } t \geq 3\delta_j r_j, \\ 0 &\leq h_j(t) \leq 1 \quad \text{and} \quad |h'_j(t)| \leq 2/(\delta_j r_j). \end{aligned}$$

We define  $p_j \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R}^n)$  and  $q_j \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R}^n)$  so that

$$\begin{aligned} p_j(x) &= T_{\mathfrak{h}}(x) + (1 - f_j(|T_{\mathfrak{h}}(x)|))h_j(|T_{\mathfrak{h}}^\perp(x)|)T_{\mathfrak{h}}^\perp(x) \\ \text{and} \quad q_j(x) &= T_{\mathfrak{h}}(x) + (1 - g_j(|T_{\mathfrak{h}}(x)|))h_j(|T_{\mathfrak{h}}^\perp(x)|)T_{\mathfrak{h}}^\perp(x) \quad \text{for } x \in \mathbf{R}^n. \end{aligned}$$

We set  $B_j = \mathbf{U}(0, 2r_j) \cap (T + \mathbf{B}(0, 2\delta_j r_j))$ . Then

$$(120) \quad \text{Lip}(p_j|_{B_j}) \leq 6 + \frac{8\delta_j}{\varepsilon_j} \leq 6 + \delta_j^{1/2} \quad \text{and} \quad \text{Lip}(q_j|_{B_j}) \leq 6 + \frac{8\delta_j}{\varepsilon_j} \leq 6 + \delta_j^{1/2}.$$

Using (119) and possibly passing to a subsequence of  $\{S_i : i \in \mathcal{P}\}$  we shall further assume that for  $i, j \in \mathcal{P}$  with  $i \geq j$  there holds

$$(121) \quad \mathcal{H}^m(S_i \cap \mathbf{U}(0, 3r_j/2) \sim (T + \mathbf{B}(0, 2\delta_j r_j))) = 0.$$

For  $j \in \mathcal{P}$  the map  $p_j$  is clearly admissible so for  $i \in \mathcal{P}$  we have  $p_j[S_i] \in \mathcal{C}$  and

$$(122) \quad \Phi_F(p_j[S_i] \cap U) \geq \mu \quad \text{and} \quad \liminf_{i \rightarrow \infty} (\Phi_F(p_j[S_i] \cap U) - \Phi_F(S_i \cap U)) \geq 0.$$

Define  $A_j = \mathbf{B}(0, r_j) \sim \mathbf{U}(0, (1 - 3\varepsilon_j)r_j)$  for  $j \in \mathcal{P}$  and  $\xi_1 = \inf \text{im } F > 0$  and  $\xi_2 = \sup \text{im } F < \infty$ . For  $i, j \in \mathcal{P}$  with  $i \geq j$ , recalling (120), we have

$$\begin{aligned} q_j[S_i] &= (S_i \sim \mathbf{B}(0, r_j)) \cup (q_j[S_i \cap A_j]) \cup (S_i \cap \mathbf{U}(0, (1 - 3\varepsilon_j)r_j)); \\ \text{thus,} \quad \Phi_F(q_j[S_i] \cap U) &= \Phi_F(S_i \cap U) + \Phi_F(q_j[S_i \cap A_j]) - \Phi_F(S_i \cap A_j) \\ &\leq \Phi_F(S_i \cap U) + \kappa_j \mathcal{H}^m(S_i \cap A_j). \end{aligned}$$

where  $\kappa_j = \xi_2(6 + \delta_j^{1/2})^m - \xi_1$  and  $\kappa_\infty = 6^m \xi_2 - \xi_1$ . In consequence

$$\limsup_{i \rightarrow \infty} (\Phi_F(q_j[S_i] \cap U) - \Phi_F(S_i \cap U)) \leq \kappa_j \limsup_{i \rightarrow \infty} \mathcal{H}^m(S_i \cap A_j).$$

Since  $\mathbf{v}(S_i \cap U) \rightarrow V \in \mathbf{V}_m(U)$  and  $A_j$  is compact, we have

$$\begin{aligned} \limsup_{i \rightarrow \infty} \mathcal{H}^m(S_i \cap A_j) &\leq \|V\|(A_j); \\ (123) \quad \text{thus,} \quad \limsup_{i \rightarrow \infty} (\Phi_F(q_j[S_i] \cap U) - \Phi_F(S_i \cap U)) &\leq \kappa_j \|V\|(A_j). \end{aligned}$$

Since  $(1 - \varepsilon_j/2)r_j + 3\delta_j r_j < r_j$ , we have

$$q_j|_{\mathbf{R}^n \sim \mathbf{U}(0, r_j)} = p_j|_{\mathbf{R}^n \sim \mathbf{U}(0, r_j)} = \text{id}_{\mathbf{R}^n \sim \mathbf{U}(0, r_j)}$$

$$\text{and } q_j[\mathbf{U}(0, r_j)] \subseteq \mathbf{U}(0, r_j), \quad p_j[\mathbf{U}(0, r_j)] \subseteq \mathbf{U}(0, r_j),$$

so we obtain for  $i, j \in \mathcal{P}$  with  $i \geq j$

$$q_j[S_i] \cap (\mathbf{R}^n \sim \mathbf{U}(0, r_j)) = S_i \cap (\mathbf{R}^n \sim \mathbf{U}(0, r_j)) = p_j[S_i] \cap (\mathbf{R}^n \sim \mathbf{U}(0, r_j))$$

$$\text{and } q_j[S_i] \cap \mathbf{U}(0, r_j) = q_j[S_i \cap \mathbf{U}(0, r_j)], \quad p_j[S_i] \cap \mathbf{U}(0, r_j) = p_j[S_i \cap \mathbf{U}(0, r_j)].$$

Hence, recalling (122) and (123), we see that for  $j \in \mathcal{P}$

$$(124) \quad \limsup_{i \rightarrow \infty} (\Phi_F(q_j[S_i \cap \mathbf{U}(0, r_j)]) - \Phi_F(p_j[S_i \cap \mathbf{U}(0, r_j)])) \leq \kappa_j \|V\|(A_j).$$

For  $i, j \in \mathcal{P}$  define

$$\tilde{p}_j = \mu_{1/r_j} \circ p_j \circ \mu_{r_j}, \quad \tilde{q}_j = \mu_{1/r_j} \circ q_j \circ \mu_{r_j}, \quad S_{j,i} = \mu_{1/r_j}[S_i],$$

$$\tilde{A}_j = \mathbf{B}(0, 1) \sim \mathbf{U}(0, 1 - 3\varepsilon_j), \quad W_{j,i} = \mu_{1/r_j} \circ q_j[S_i], \quad Z_{j,i} = \mu_{1/r_j} \circ p_j[S_i],$$

$$F_j = \mu_{r_j}^\# F, \quad \text{i.e., } F_j(x, T) = r_j^m F(r_j x, T) \quad \text{for } (x, T) \in \mathbf{R}^n \times \mathbf{G}(n, m).$$

Then

$$\mu_{1/r_j}[p_j[S_i] \cap \mathbf{U}(0, r_j)] = Z_{j,i} \cap \mathbf{U}(0, 1), \quad \mu_{1/r_j}[q_j[S_i] \cap \mathbf{U}(0, r_j)] = W_{j,i} \cap \mathbf{U}(0, 1),$$

$$(125) \quad \Phi_{F_j}(X) = \Phi_F((\mu_{r_j})_\# X) \quad \text{for } X \in \mathbf{V}_m(\mathbf{R}^n).$$

Since  $\mathbf{v}(\mu_{1/r_j}[S_i]) = (\mu_{1/r_j})_\# \mathbf{v}(S_i)$  we get for  $j \in \mathcal{P}$  using (124)

$$\limsup_{i \rightarrow \infty} r_j^{-m} (\Phi_{F_j}(W_{j,i} \cap \mathbf{U}(0, 1)) - \Phi_{F_j}(Z_{j,i} \cap \mathbf{U}(0, 1))) \leq r_j^{-m} \kappa_j \|V\|(A_j)$$

$$= \kappa_j \|(\mu_{1/r_j})_\# V\|(\tilde{A}_j);$$

hence, recalling  $\Theta^m(\|V\|, x_0) \in (0, \infty)$  and [All72, 3.4(2)],

$$\limsup_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} r_j^{-m} (\Phi_{F_j}(W_{j,i} \cap \mathbf{U}(0, 1)) - \Phi_{F_j}(Z_{j,i} \cap \mathbf{U}(0, 1)))$$

$$\leq \kappa_\infty \|C\|(\text{Bdry } \mathbf{U}(0, 1)) = 0.$$

Employing (121) we obtain for  $i, j \in \mathcal{P}$  with  $i \geq j$

$$\mathcal{H}^m(S_{j,i} \cap \mathbf{U}(0, 3/2) \sim (T + \mathbf{B}(0, 2\delta_j))) = 0.$$

For  $j \in \mathcal{P}$  define  $\rho_j = 1 - 3\varepsilon_j/2$ . We will now check that if  $i, j \in \mathcal{P}$  satisfy  $i \geq j$  and are big enough, then one cannot deform  $W_{j,i} \cap \mathbf{B}(0, \rho_j)$  onto  $T \cap \text{Bdry } \mathbf{U}(0, \rho_j)$  by any map from  $\mathfrak{D}(\mathbf{U}(0, \rho_j))$ . Assume, by contradiction, that there exists  $\phi \in \mathfrak{D}(\mathbf{U}(0, \rho_j))$  such that  $\phi[W_{j,i} \cap \mathbf{B}(0, \rho_j)] \subseteq T \cap \text{Bdry } \mathbf{U}(0, \rho_j)$ . Clearly  $\mu_{r_j} \circ \phi[W_{j,i}] \in \mathcal{C}$  so

$$(126) \quad \mu \leq \Phi_F(\mu_{r_j} \circ \phi[W_{j,i}] \cap U) = \Phi_F(\mu_{r_j}[W_{j,i}] \cap U) - \Phi_F(\mu_{r_j}[W_{j,i} \cap \mathbf{U}(0, \rho_j)])$$

$$= \Phi_F(q_j[S_i] \cap U) - \Phi_F(q_j[S_i] \cap \mathbf{U}(0, r_j \rho_j))$$

$$\leq \Phi_F(q_j[S_i] \cap U) - \Phi_F(q_j[S_i] \cap \mathbf{U}(0, (1 - 3\varepsilon_j)r_j))$$

$$= \Phi_F(S_i \cap U) + (\Phi_F(q_j[S_i] \cap U) - \Phi_F(S_i \cap U)) - \Phi_F(S_i \cap \mathbf{U}(0, (1 - 3\varepsilon_j)r_j)).$$

We choose  $j_0 \in \mathcal{P}$  so big that for  $j \geq j_0$  we have

$$(127) \quad r_j^{-m} \xi_1 \|V\| \mathbf{U}(0, (1 - 3\varepsilon_j)r_j) - r_j^{-m} 2\kappa_j \|V\|(A_j) > 2^{-4} \xi_1 \Theta^m(\|V\|, 0),$$

which is possible because  $\lim_{j \rightarrow \infty} r_j^{-m} \kappa_j \|V\|(A_j) = 0$  and  $\lim_{j \rightarrow \infty} r_j^{-m} \|V\| \mathbf{U}(0, (1 - 3\varepsilon_j)r_j) = \Theta^m(\|V\|, 0) > 0$ . For each  $j \geq j_0$  we select  $i_0 = i_0(j) \in \mathcal{P}$  such that  $i_0 \geq j$  and for  $i \geq i_0$

$$(128) \quad \begin{aligned} \Phi_F(S_i \cap U) - \mu &< 2^{-7} r_j^m \xi_1 \Theta^m(\|V\|, 0) \\ \text{and } \Phi_F(q_j[S_i] \cap U) - \Phi_F(S_i \cap U) &\leq 2\kappa_j \|V\|(A_j), \end{aligned}$$

which is possible because  $\{S_i : i \in \mathcal{P}\}$  is a minimising sequence and due to (123). Combining (126), (127), and (128) we get for  $j \geq j_0$  and  $i \geq i_0(j)$  the following contradictory estimate

$$\mu \leq \mu + r_j^m \xi_1 \Theta^m(\|V\|, 0) (2^{-7} - 2^{-4}) < \mu.$$

We now know that  $W_{j,i}$  cannot be deformed onto  $T \cap \text{Bdry } \mathbf{U}(0, \rho_j)$  by any admissible map from  $\mathfrak{D}(\mathbf{U}(0, \rho_j))$  given  $j \geq j_0$  and  $i \geq i_0(j)$ . As a consequence we can make use of ellipticity of  $F$  and we observe that

$$(129) \quad T_i[W_{j,i} \cap \mathbf{B}(0, 1)] \cap \mathbf{B}(0, \rho_j) = Z_{j,i} \cap \mathbf{B}(0, \rho_j) = \tilde{p}_j[W_{j,i}] \cap \mathbf{B}(0, \rho_j) = T \cap \mathbf{B}(0, \rho_j)$$

because otherwise we could deform  $W_{j,i}$  onto  $T \cap \text{Bdry } \mathbf{U}(0, \rho_j)$ . In particular,

$$\liminf_{j \rightarrow \infty} \liminf_{i \rightarrow \infty} \mathcal{H}^m(W_{j,i} \cap \mathbf{B}(0, 1)) \geq \lim_{j \rightarrow \infty} \mathcal{H}^m(T \cap \mathbf{B}(0, \rho_j)) = \alpha(m).$$

Define  $X_j = \mathbf{U}(0, 1) \sim \mathbf{B}(0, \rho_j)$ . We know that there is a constant  $\xi_3 > 0$  such that for  $i, j \in \mathcal{P}$  with  $i \geq i_0(j)$  and  $j \geq j_0$

$$\begin{aligned} 0 &\leq \mathcal{H}^m(W_{j,i} \cap \mathbf{U}(0, \rho_j)) - \mathcal{H}^m(T \cap \mathbf{U}(0, \rho_j)) \\ &\leq \xi_3 r_j^{-m} (\Phi_{F^0}(\mu_{r_j}[W_{j,i} \cap \mathbf{U}(0, \rho_j)]) - \Phi_{F^0}(\mu_{r_j}[T \cap \mathbf{U}(0, \rho_j)])) \\ &\leq \xi_3 r_j^{-m} (\Phi_{F_j^0}(W_{j,i} \cap \mathbf{U}(0, 1)) - \Phi_{F_j^0}(Z_{j,i} \cap \mathbf{U}(0, 1))) + r_j^{-m} \xi_3 \Phi_{F_j^0}(Z_{j,i} \cap X_j). \end{aligned}$$

Since  $\tilde{q}_j(\mathbf{U}(0, 1)) \subseteq \mathbf{U}(0, 1)$  and  $\tilde{q}_j(x) = x$  for  $x \in \mathbf{U}(0, 1 - 3\varepsilon_j) \cup (\mathbf{R}^n \sim \mathbf{U}(0, 1))$ , we see that

$$W_{j,i} \cap \mathbf{U}(0, 1) \supseteq (S_{j,i} \cap \mathbf{B}(0, 1 - 3\varepsilon_j)) \cup (W_{j,i} \cap \tilde{A}_j);$$

$$\text{thus, } \mathcal{H}^m(W_{j,i} \cap \mathbf{U}(0, 1)) \geq \mathcal{H}^m(S_{j,i} \cap \mathbf{U}(0, 1)) - \mathcal{H}^m(S_{j,i} \cap \tilde{A}_j) + \mathcal{H}^m(W_{j,i} \cap \tilde{A}_j).$$

Hence, we get

$$\begin{aligned} (130) \quad &|\mathcal{H}^m(S_{j,i} \cap \mathbf{U}(0, 1)) - \mathcal{H}^m(T \cap \mathbf{U}(0, 1))| \\ &\leq |\mathcal{H}^m(W_{j,i} \cap \mathbf{U}(0, 1)) - \mathcal{H}^m(T \cap \mathbf{U}(0, 1))| + \mathcal{H}^m(S_{j,i} \cap \tilde{A}_j) + \mathcal{H}^m(W_{j,i} \cap \tilde{A}_j) \\ &\leq (\mathcal{H}^m(W_{j,i} \cap \mathbf{U}(0, \rho_j)) - \mathcal{H}^m(T \cap \mathbf{U}(0, \rho_j))) + \mathcal{H}^m(S_{j,i} \cap \tilde{A}_j) + 2\mathcal{H}^m(W_{j,i} \cap \tilde{A}_j) \\ &\quad + \mathcal{H}^m(T \cap \mathbf{U}(0, 1) \sim \mathbf{U}(0, \rho_j)) \\ &\leq \xi_3 r_j^{-m} (\Phi_{F_j^0}(W_{j,i} \cap \mathbf{U}(0, 1)) - \Phi_{F_j^0}(Z_{j,i} \cap \mathbf{U}(0, 1))) + r_j^{-m} \xi_3 \Phi_{F_j^0}(Z_{j,i} \cap X_j) \\ &\quad + 2\mathcal{H}^m(W_{j,i} \cap \tilde{A}_j) + \mathcal{H}^m(S_{j,i} \cap \tilde{A}_j) + \mathcal{H}^m(T \cap \mathbf{U}(0, 1) \sim \mathbf{U}(0, \rho_j)). \end{aligned}$$

Recalling (120) and (125) we see that

$$(131) \quad r_j^{-m} \xi_3 \Phi_{F_j^0}(Z_{j,i} \cap X_j) \leq \xi_3 \xi_2 (6 + \delta_j^{1/2})^m \mathcal{H}^m(S_{j,i} \cap \tilde{A}_j),$$

$$(132) \quad \mathcal{H}^m(W_{j,i} \cap \tilde{A}_j) \leq (6 + \delta_j^{1/2})^m \mathcal{H}^m(S_{j,i} \cap \tilde{A}_j).$$

We observe that

$$(133) \quad \limsup_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} \mathcal{H}^m(S_{j,i} \cap \tilde{A}_j) \leq \limsup_{j \rightarrow \infty} r_j^{-m} \|V\|(\tilde{A}_j) \\ = \limsup_{j \rightarrow \infty} \|(\mu_{1/r_j})_{\#} V\|(\tilde{A}_j) \leq \|C\|(\text{Bdry } \mathbf{B}(0, 1)) = 0.$$

Let us define  $\omega : (0, \infty) \rightarrow \mathbf{R}$  by the formula

$$\omega(r) = \sup\{|F(0, T) - F(x, T)| : x \in \mathbf{B}(0, r), T \in \mathbf{G}(n, m)\}.$$

Then, we may write

$$(134) \quad r_j^{-m} (\Phi_{F_j^0}(W_{j,i} \cap \mathbf{U}(0, 1)) - \Phi_{F_j^0}(Z_{j,i} \cap \mathbf{U}(0, 1))) \\ \leq r_j^{-m} (\Phi_{F_j}(W_{j,i} \cap \mathbf{U}(0, 1)) - \Phi_{F_j}(Z_{j,i} \cap \mathbf{U}(0, 1))) + \omega(r_j) \mathcal{H}^m(W_{j,i} \cap \mathbf{U}(0, 1)).$$

Using again (120) we have

$$(135) \quad \limsup_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} \mathcal{H}^m(W_{j,i} \cap \mathbf{U}(0, 1)) \\ \leq \limsup_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} (6 + \delta_j^{1/2})^m \mathcal{H}^m(S_{j,i} \cap \mathbf{U}(0, 1)) \leq \|C\| \mathbf{B}(0, 1) < \infty.$$

Since  $F$  is of class  $\mathcal{C}^0$  we see that  $\lim_{r \rightarrow 0} \omega(r) = 0$ ; hence, combining (130) with (131), (132), (133), (134), and (135) we obtain

$$(136) \quad \limsup_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} |\mathcal{H}^m(S_{j,i} \cap \mathbf{U}(0, 1)) - \mathcal{H}^m(T \cap \mathbf{U}(0, 1))| \\ \leq \limsup_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} \xi_3 r_j^{-m} (\Phi_{F_j}(W_{j,i} \cap \mathbf{U}(0, 1)) - \Phi_{F_j}(Z_{j,i} \cap \mathbf{U}(0, 1))).$$

For any  $i, j \in \mathcal{P}$  we have

$$(137) \quad \Phi_F(\mu_{r_j}[Z_{j,i}] \cap U) = \Phi_F(S_i \cap U) + (\Phi_{F_j}(Z_{j,i} \cap \mathbf{U}(0, 1)) - \Phi_{F_j}(W_{j,i} \cap \mathbf{U}(0, 1))) \\ + \Phi_{F_j}(W_{j,i} \cap \tilde{A}_j) - \Phi_{F_j}(S_{j,i} \cap \tilde{A}_j).$$

Since  $V = \lim_{i \rightarrow \infty} \mathbf{v}(S_i \cap U)$  is minimising, i.e.,  $\Phi_F(V) = \mu$  we have

$$\Phi_F(V) \leq \Phi_F(\mu_{r_j}[Z_{j,i}] \cap U) \quad \text{for each } i, j \in \mathcal{P};$$

hence, transforming (137) we get

$$r_j^{-m} (\Phi_{F_j}(W_{j,i} \cap \mathbf{U}(0, 1)) - \Phi_{F_j}(Z_{j,i} \cap \mathbf{U}(0, 1))) \leq r_j^{-m} (\Phi_F(S_i \cap U) - \Phi_F(V)) \\ + r_j^{-m} \Phi_{F_j}(W_{j,i} \cap \tilde{A}_j) + r_j^{-m} \Phi_{F_j}(S_{j,i} \cap \tilde{A}_j)$$

Estimating as in (131), (132), (133) we reach the conclusion

$$\limsup_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} r_j^{-m} (\Phi_{F_j}(W_{j,i} \cap \mathbf{U}(0,1)) - \Phi_{F_j}(Z_{j,i} \cap \mathbf{U}(0,1))) = 0.$$

Plugging this into (136) we obtain

$$\begin{aligned} 0 &= \limsup_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} |\mathcal{H}^m(S_{j,i} \cap \mathbf{U}(0,1)) - \mathcal{H}^m(T \cap \mathbf{U}(0,1))| \\ &= \lim_{j \rightarrow \infty} r_j^{-m} \|V\| \mathbf{U}(0, r_j) - \alpha(m) = \alpha(m) (\Theta^m(\|V\|, 0) - 1). \end{aligned}$$

Hence,  $\|C\| = \mathcal{H}^m \llcorner T$  by [All72, 3.4(2)]. To have  $C = \mathbf{v}(T)$  we still need to show that  $P = T$  for  $C$  almost all  $(x, P)$ .

We need to employ the area formula to  $S_{j,i}$  so we first need to prove the following claim. For each  $i, j \in \mathcal{P}$  decompose  $S_{j,i} \cap \mathbf{B}(0,1)$  into a disjoint sum  $S_{j,i} \cap \mathbf{B}(0,1) = \bar{S}_{j,i} \cup \hat{S}_{j,i}$ , where  $\bar{S}_{j,i}$  is purely  $(\mathcal{H}^m, m)$  unrectifiable and  $\hat{S}_{j,i}$  is  $(\mathcal{H}^m, m)$  rectifiable. We already proved that

$$(138) \quad \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \mathcal{H}^m(S_{j,i} \cap \mathbf{B}(0,1)) = \alpha(m) \Theta^m(\|V\|, 0) = \alpha(m).$$

We claim that

$$(139) \quad \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \mathcal{H}^m(\bar{S}_{j,i} \cap \mathbf{B}(0,1)) = 0.$$

Assume it is not true. Then, possibly passing to a sub-sequence, there exists  $\delta > 0$  such that

$$\forall j \in \mathcal{P} \exists i_0 = i_0(j) \forall i \geq i_0 \quad \delta < \mathcal{H}^m(\bar{S}_{j,i} \cap \mathbf{B}(0,1)) < 2\delta.$$

Let  $\beta(n)$  be the optimal constant in the Besicovitch-Federer covering theorem [Fed69, 2.8.14]. Choose  $\iota \in (0, 2^{-100})$  so that  $2\iota\alpha(m)\xi_2\Gamma_{10.4}\beta(n) < 2^{-100}\xi_1\delta$ . Define

$$\bar{\mathcal{B}}_{j,i} = \left\{ \mathbf{B}(x, t) : \begin{array}{l} t \in (0, \iota), x \in \bar{S}_{j,i}, \mathcal{H}^m(S_{j,i} \cap \text{Bdry } \mathbf{B}(x, t)) = 0, \\ \mathcal{H}^m(\hat{S}_{j,i} \cap \mathbf{B}(x, t)) \leq \iota \mathcal{H}^m(S_{j,i} \cap \mathbf{B}(x, t) \cap \mathbf{B}(0,1)), \\ \mathcal{H}^m(\bar{S}_{j,i} \cap \mathbf{B}(x, t)) \geq (1 - \iota) \mathcal{H}^m(S_{j,i} \cap \mathbf{B}(x, t) \cap \mathbf{B}(0,1)) \end{array} \right\}.$$

Employ the Besicovitch-Federer covering theorem [Fed69, 2.8.14] to obtain at most countable subfamily  $\mathcal{B}_{j,i} \subseteq \bar{\mathcal{B}}_{j,i}$  such that

$$\bar{S}_{j,i} \subseteq \bigcup \mathcal{B}_{j,i} \cap \mathbf{B}(0,1) \quad \text{and} \quad \mathbf{1}_{\bigcup \mathcal{B}_{j,i}} \leq \sum_{B \in \mathcal{B}_{j,i}} \mathbf{1}_B \leq \beta(n) \mathbf{1}_{\bigcup \mathcal{B}_{j,i}}.$$

Define  $E_{j,i} = \bigcup \mathcal{B}_{j,i} \cap \mathbf{B}(0,1)$  and observe that

$$\mathcal{H}^m(\hat{S}_{j,i} \cap E_{j,i}) \leq \varepsilon \sum_{B \in \mathcal{B}_{j,i}} \mathcal{H}^m(S_{j,i} \cap B \cap \mathbf{B}(0,1)) \leq \varepsilon \beta(n) \mathcal{H}^m(S_{j,i} \cap E_{j,i}).$$

Employ 10.4 with  $\iota$  in place of  $\varepsilon$  to obtain the map  $f_{j,i} \in \mathcal{D}(\text{Int } E_{j,i})$  of class  $\mathcal{C}^\infty$  such that

$$\begin{aligned} \mathcal{H}^m(f_{j,i}[E_{j,i} \cap \bar{S}_{j,i}]) &\leq \iota \mathcal{H}^m(E_{j,i} \cap \bar{S}_{j,i}) \leq 2\iota\delta \\ \text{and } \mathcal{H}^m(f_{j,i}[E_{j,i} \cap \hat{S}_{j,i}]) &\leq \Gamma_{10.4} \mathcal{H}^m(E_{j,i} \cap \hat{S}_{j,i}) \leq \iota \Gamma_{10.4} \beta(n) \mathcal{H}^m(S_{j,i} \cap \mathbf{B}(0,1)). \end{aligned}$$

Recall (138) and choose  $j, i_1 \in \mathcal{P}$  such that  $i_i \geq i_0(j)$  and  $\mathcal{H}^m(S_{j,i} \cap \mathbf{B}(0, 1)) < 2\alpha(m)$  for all  $i \geq i_1$ . We estimate

$$\begin{aligned} \limsup_{i \rightarrow \infty} \Phi_F(\mu_{r_j}[f_{j,i}[S_{j,i}] \cap U]) &= \limsup_{i \rightarrow \infty} \Phi_{F_j}(f_{j,i}[S_{j,i}] \cap \mu_{1/r_j}[U]) \\ &\leq \limsup_{i \rightarrow \infty} (\Phi_F(S_i \cap U) - \Phi_{F_j}(\bar{S}_{j,i} \cap E_{j,i}) + \Phi_{F_j}(f_{j,i}[\hat{S}_{j,i} \cap E_{j,i}]) + \Phi_{F_j}(f_{j,i}[\bar{S}_{j,i} \cap E_{j,i}])) \\ &\leq \limsup_{i \rightarrow \infty} (\Phi_F(S_i \cap U) + r_j^m(-\xi_1\delta + 2\iota\alpha(m)\xi_2\Gamma_{10.4}\beta(n) + 2\iota\delta)) < \lim_{i \rightarrow \infty} \Phi_F(S_i \cap U) = \mu. \end{aligned}$$

This contradicts the definition of  $\mu$ , so the claim is proven.

Now we see that

$$\lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \mathcal{H}^m(\hat{S}_{j,i} \cap \mathbf{B}(0, 1)) = \alpha(m)\Theta^m(\|V\|, 0) = \alpha(m).$$

Set  $p_{j,i} = T_{\mathfrak{h}}|_{\hat{S}_{j,i}}$ . Since  $p_{j,i}(x) = \tilde{p}_j(x)$  for  $x \in S_{j,i} \cap \mathbf{B}(0, 1) \sim \tilde{A}_j$  and recalling (133) and (129) we see that

$$\lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \mathcal{H}^m(p_{j,i}[\hat{S}_{j,i} \cap \mathbf{B}(0, 1)]) = \alpha(m).$$

Clearly  $\text{Lip}(p_{j,i}) \leq 1$  so  $1 - \text{ap } J_m p_{j,i}(x) \geq 0$  for  $\mathcal{H}^m$  almost all  $x \in \hat{S}_{j,i}$ . Employing the area formula [Fed69, 3.2.20] we have

$$\begin{aligned} (140) \quad 0 &\leq \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \int_{\hat{S}_{j,i} \cap \mathbf{B}(0, 1)} 1 - \text{ap } J_m p_{j,i} \, d\mathcal{H}^m \\ &= \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \mathcal{H}^m(\hat{S}_{j,i} \cap \mathbf{B}(0, 1)) - \int_{p_{j,i}[\hat{S}_{j,i} \cap \mathbf{B}(0, 1)]} \mathcal{H}^0(p_{j,i}^{-1}\{y\}) \, d\mathcal{H}^m(y) \\ &\leq \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} (\mathcal{H}^m(\hat{S}_{j,i} \cap \mathbf{B}(0, 1)) - \mathcal{H}^m(p_{j,i}[\hat{S}_{j,i} \cap \mathbf{B}(0, 1)])) = 0. \end{aligned}$$

Hence, employing 11.3, we have

$$\lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \int_{\mathbf{B}(0, 1)} \|P_{\mathfrak{h}} - T_{\mathfrak{h}}\|^2 \, d\mathbf{v}(\hat{S}_{j,i})(x, P) = 0.$$

Define  $\varphi(x, P) = \mathbb{1}_{\mathbf{B}(0, 1)}(x) \|S_{\mathfrak{h}} - T_{\mathfrak{h}}\|^2$  for  $(x, P) \in \mathbf{R}^n \times \mathbf{G}(n, m)$ . Recalling (139) and noting that  $\|C\|(\text{Bdry } \mathbf{B}(0, 1)) = 0$ , we see that

$$0 = \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \mathbf{v}(S_{j,i})(\varphi) = C(\varphi) = \int_{\mathbf{B}(0, 1)} \|P_{\mathfrak{h}} - T_{\mathfrak{h}}\|^2 \, dC(x, P).$$

Therefore,  $P = T$  for  $C$  almost all  $(x, P) \in \mathbf{B}(0, 1) \times \mathbf{G}(n, m)$ .

Now, since  $C \in \text{VarTan}(V, 0)$  was chosen arbitrarily we obtain for all  $C \in \text{VarTan}(V, 0)$

$$(141) \quad C(\psi) = \int_T \psi(x, T) \, d\mathcal{H}^m(x) \quad \text{whenever } \psi \in \mathcal{H}(\mathbf{U}(0, 1) \times \mathbf{G}(n, m)).$$

In particular, if  $C \in \text{VarTan}(V, 0)$  and  $\rho \in (0, 1)$ , then  $C' = (\mu_\rho)_\# C \in \text{VarTan}(V, 0)$  also satisfies (141); hence, for all  $C \in \text{VarTan}(V, 0)$

$$C(\psi) = \int_T \psi(x, T) \, d\mathcal{H}^m(x) \quad \text{whenever } \psi \in \mathcal{H}(\mathbf{R}^n \times \mathbf{G}(n, m)). \quad \square$$

Now we have all the ingredients to prove our main theorem.



*Proof of 3.16.* We first apply 11.2 to obtain a minimising sequence  $\{S_i : i \in \mathcal{P}\} \subseteq \mathcal{C}$ , and  $V \in \mathbf{V}_m(U)$ , and  $S \in \mathcal{C}$  satisfying

$$\begin{aligned} \lim_{i \rightarrow \infty} \Phi_F(S_i \cap U) = \mu \quad \text{and} \quad V = \lim_{i \rightarrow \infty} \mathbf{v}(S_i \cap U) \quad \text{and} \quad \mathcal{H}^m(S \cap U \sim \text{spt } \|V\|) = 0 \\ \text{and} \quad \lim_{i \rightarrow \infty} d_{\mathcal{H},K}(S \cap U, S_i \cap U) = 0 \quad \text{for } K \subseteq U \text{ compact.} \end{aligned}$$

Next, we employ 10.1 to see that  $\text{spt } \|V\|$  is countably  $(\mathcal{H}^m, m)$  rectifiable and has locally finite  $\mathcal{H}^m$  measure. In particular,  $\Theta^m(\mathcal{H}^m \llcorner \text{spt } \|V\|, x) = 1$  for  $\mathcal{H}^m$  almost all  $x \in \text{spt } \|V\| \subseteq U$  so, using 9.4, also the density  $\Theta^m(\|V\|, x)$  exists and is finite for  $\|V\|$  almost all  $x \in U$ . After that, we recall 10.3 to argue that for  $\mathcal{H}^m$  almost all  $x \in \text{spt } \|V\|$  the classical tangent cone  $\text{Tan}(\text{spt } \|V\|, x)$  is an  $m$ -dimensional subspace of  $\mathbf{R}^n$ . Having made all these observations we apply 11.4 to see that  $\Theta^m(\|V\|, x) = 1$  and  $\text{Tan}(\text{spt } \|V\|, x) = T$  for  $V$  almost all  $(x, T) \in U \times \mathbf{G}(n, m)$ . We know  $\mathcal{H}^m(S \cap U \sim \text{spt } \|V\|) = 0$ , which means that  $V = \mathbf{v}(S \cap U)$  and that  $S$  and  $\{S_i : i \in \mathcal{P}\}$  satisfy all the conditions of 3.16.  $\square$

## 12 An example of a good class: homological spanning

Let us fix an abelian coefficient group  $G$ . We shall work in the category  $\mathcal{A}_1$  of *arbitrary pairs and their maps* as defined in [ES52, I,1]. This means that  $(X, A)$  is an object in  $\mathcal{A}_1$  if  $X$  is a topological space and  $A \subseteq X$  is an arbitrary subset with the relative topology. Morphisms in  $\mathcal{A}_1$  between  $(X, A)$  and  $(Y, B)$  are continuous functions  $f : X \rightarrow Y$  such that  $f[A] \subseteq B$ . If  $(X, A), (Y, B)$  are objects in  $\mathcal{A}_1$  and  $f : (X, A) \rightarrow (Y, B)$  is a morphism in  $\mathcal{A}_1$ , then the symbol  $\check{\mathbf{H}}_k(X, A; G)$  shall denote the  $k^{\text{th}}$  Čech homology group [ES52, IX,3.3] of the pair  $(X, A)$  with coefficients in  $G$  and  $\check{\mathbf{H}}_k(f) : \check{\mathbf{H}}_k(X, A; G) \rightarrow \check{\mathbf{H}}_k(Y, B; G)$  the induced morphism of abelian groups. In case  $A = \emptyset$ , we write  $\check{\mathbf{H}}_k(X; G) = \check{\mathbf{H}}_k(X, \emptyset; G)$ . For any sets  $X \subseteq Y \subseteq \mathbf{R}^n$ , we will denote by  $i_{X,Y}$  the inclusion map  $X \hookrightarrow Y$ .

**12.1 Definition.** Let  $B$  be a closed subset of  $\mathbf{R}^n$ ,  $L$  a subgroup of  $\check{\mathbf{H}}_{m-1}(B; G)$ . We say that a closed set  $E \subseteq \mathbf{R}^n$  spans  $L$  if  $L \subseteq \ker(\check{\mathbf{H}}_{m-1}(i_{B, B \cup E}))$ . In other words,  $E$  spans  $L$  if the composition

$$L \hookrightarrow \check{\mathbf{H}}_{m-1}(B; G) \xrightarrow{\check{\mathbf{H}}_{m-1}(i_{B, B \cup E})} \check{\mathbf{H}}_{m-1}(B \cup E; G)$$

is zero.

We denote by  $\check{\mathcal{C}}(B, L, G)$  the collection of all closed subsets of  $\mathbf{R}^n$  which span  $L$ .

We shall prove that  $\check{\mathcal{C}}(B, L, G)$  is a good class in the sense of definition 3.4.

**12.2 Lemma.** Let  $B$  be a closed subset of  $\mathbf{R}^n$ ,  $L$  a subgroup of  $\check{\mathbf{H}}_{m-1}(B; G)$ . Let  $\{E_k \subseteq \mathbf{R}^n : k \in \mathcal{P}\}$  be a decreasing sequence of closed sets. If  $B \subseteq E_{k+1} \subseteq E_k$  and  $L \subseteq \ker(\check{\mathbf{H}}_{m-1}(i_{B, E_k}))$  for all  $k \geq 1$ , then, by setting  $E = \bigcap_{k \geq 1} E_k$ , we have  $L \subseteq \ker(\check{\mathbf{H}}_{m-1}(i_{B, E}))$

*Proof.* Since  $E = \bigcap_{k \geq 1} E_k$ , we have that  $E = \varprojlim E_k$ , see for example [ES52, Theorem 2.5 on p. 260]. We let

$$\varphi : \check{\mathbf{H}}_{m-1}(E; G) \rightarrow \varprojlim \check{\mathbf{H}}_{m-1}(E_k; G)$$

be the natural isomorphism, and let

$$\pi_k : \varprojlim \check{\mathbf{H}}_{m-1}(E_k; G) \rightarrow \check{\mathbf{H}}_{m-1}(E_k; G)$$

be the natural projections. Then

$$\check{\mathbf{H}}_{m-1}(i_{E,E_k}) = \varphi \circ \pi_k.$$

Since

$$\check{\mathbf{H}}_{m-1}(i_{B,E_j}) = \check{\mathbf{H}}_{m-1}(i_{E_k,E_j}) \circ \check{\mathbf{H}}_{m-1}(i_{B,E}),$$

by the universal property of inverse limit, there exist a homomorphism

$$\psi : \check{\mathbf{H}}_{m-1}(B; G) \rightarrow \varprojlim \check{\mathbf{H}}_{m-1}(E_k; G)$$

such that

$$\check{\mathbf{H}}_{m-1}(i_{B,E_k}) = \pi_k \circ \psi.$$

Then

$$\pi_k \circ \psi(L) = \check{\mathbf{H}}_{m-1}(i_{B,E_k})(L) = 0,$$

thus  $\psi(L) = 0$ . We see that

$$\check{\mathbf{H}}_{m-1}(i_{B,E}) = \varphi^{-1} \circ \psi,$$

thus

$$\check{\mathbf{H}}_{m-1}(i_{B,E})(L) = \varphi^{-1} \circ \psi(L) = 0. \quad \square$$

**12.3 Remark.** For any continuous map  $\varphi : \mathbf{R}^n \rightarrow \mathbf{R}^n$  with  $\varphi|_B = \text{id}_B$ , we have

$$\varphi[S] \in \check{\mathcal{C}}(B, L, G) \quad \text{whenever } S \in \check{\mathcal{C}}(B, L, G).$$

In particular,  $\check{\mathcal{C}}(B, L, G)$  satisfies condition (c) of definition 3.4.

**12.4 Lemma.** Let  $B, L$  and  $\check{\mathcal{C}}(B, L, G)$  be given as in Definition 12.1. Then  $\check{\mathcal{C}}(B, L, G)$  is a good class in the sense of 3.4.

*Proof.* Observe that  $\mathbf{R}^n \in \check{\mathcal{C}}(B, L, G)$  so  $\check{\mathcal{C}}(B, L, G)$  is non-empty and contains only closed sets by definition. Recalling 12.3 we only need to check condition (d) of definition 3.4.

If  $\{S_k : k \in \mathcal{P}\} \in \check{\mathcal{C}}(B, L, G)$  is a sequence such that  $S_i \rightarrow S$  locally in Hausdorff distance for some closed set  $S$ , then by putting

$$E_k = \text{Clos}(B \cup \bigcup_{i \geq k} S_i) = B \cup S \cup \bigcup_{i \geq k} S_i,$$

we have that

$$L \subseteq \ker \check{\mathbf{H}}_{m-1}(i_{B,E_k}).$$

By Lemma 12.2, we get that  $L \subseteq \ker \check{\mathbf{H}}_{m-1}(i_{B,B \cup S})$ .  $\square$

**12.5 Remark.** Replacing Čech homology group with Čech cohomology group, we let  $L$  be a subgroup of  $\check{\mathbf{H}}^{m-1}(B; G)$ ,  $\mathcal{C}$  a collection of closed sets  $E$  such that the composition

$$\check{\mathbf{H}}^{m-1}(B \cup E; G) \xrightarrow{\check{\mathbf{H}}^{m-1}(i_{B,B \cup E})} \check{\mathbf{H}}^{m-1}(B; G) \twoheadrightarrow \check{\mathbf{H}}^{m-1}(B; G)/L$$

is zero. By continuity of Čech cohomology theory, see for example [ES52, Theorem 3.1 in p. 261], we get also that  $\mathcal{C}$  is a good class. Indeed, we have a similar result as Lemma 12.2, but with Čech cohomology groups instead of Čech homology groups. For the proof we refer the reader to [Spa87, Proposition (2.7)].

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